

L^2 -SERRE DUALITY ON SINGULAR COMPLEX SPACES AND RATIONAL SINGULARITIES

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ABSTRACT. In the present paper, we devise a version of topological L^2 -Serre duality for singular complex spaces with arbitrary singularities. This duality is used to deduce various new L^2 -vanishing theorems for the $\bar{\partial}$ -equation on singular spaces. It is shown that complex spaces with rational singularities behave quite tame with respect to the $\bar{\partial}$ -equation in the L^2 -sense. More precisely: a singular point is rational if and only if the L^2 - $\bar{\partial}$ -complex is exact in this point. So, we obtain an L^2 - $\bar{\partial}$ -resolution of the structure sheaf in rational singular points.

1. INTRODUCTION

The Cauchy-Riemann operator $\bar{\partial}$ plays a fundamental role in Complex Analysis and Complex Geometry. On complex manifolds, functions – or more generally distributions – are holomorphic if and only if they are in the kernel of the $\bar{\partial}$ -operator, and the same holds in a certain sense on normal complex spaces. For forms of arbitrary degree, the importance of the $\bar{\partial}$ -operator appears strikingly for example in the notion of $\bar{\partial}$ -cohomology which can be used to represent the cohomology of complex manifolds by the Dolbeault isomorphism. Solving $\bar{\partial}$ -equations, i.e., $\bar{\partial}$ -vanishing theorems, play a central role in a vast number of problems in Complex Analysis and Geometry, let us just mention the Cousin problems.

However, a huge part of the $\bar{\partial}$ -theory is still restricted to the smooth setting. In the present paper, we introduce a version of topological L^2 -Serre duality for $\bar{\partial}$ -cohomology classes on singular complex spaces with arbitrary singularities, which allows to deduce some new L^2 -vanishing theorems for the $\bar{\partial}$ -operator on singular spaces. We obtain an explicit L^2 - $\bar{\partial}$ -resolution of the structure sheaf of spaces with rational singularities (and an L^2 - $\bar{\partial}$ -characterization of rationality).

The L^2 -theory for the $\bar{\partial}$ -operator is of particular importance in Complex Analysis and Geometry and has become indispensable for the subject after the fundamental work of Hörmander on L^2 -estimates and existence theorems for the $\bar{\partial}$ -operator [H1] and the related work of Andreotti and Vesentini [AV]. Important applications of the L^2 -theory are e.g. the Ohsawa-Takegoshi extension theorem [OT], Siu's analyticity of the level sets of Lelong numbers [S2] or the invariance of plurigenera [S3] – just to name some.

On the other hand, it is almost dispensable to mention the importance of Serre duality at all. It is one of the most important tools in Complex Algebraic Geometry and Complex Analysis. On singular spaces, the classical Serre duality has to be replaced by the more involved Grothendieck duality (developed by Ramis and Ruget in the analytic setting [RR]). In the present paper, we propose a version of topological L^2 -Serre duality which is well-adapted to singular spaces.

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The first problem one has to face when studying the $\bar{\partial}$ -equation on singular spaces is that it is not clear what kind of differential forms and operators one should consider. Recently, there has been some progress by different approaches.

Andersson and Samuelsson developed in [AS] Koppelman integral formulas for the $\bar{\partial}$ -equation on arbitrary singular complex spaces which allow for a $\bar{\partial}$ -resolution of the structure sheaf in terms of certain fine sheaves of currents, called \mathcal{A} -sheaves. These \mathcal{A} -sheaves are defined by an iterative procedure of repeated application of singular integral operators.

A second, somewhat more explicit approach is as follows: consider differential forms which are defined on the regular part of a singular variety and which are square-integrable up to the singular set. This setting seems to be quite fruitful and has some history by now (see [PS]). Also in this direction, some progress has been made recently. Øvrelid–Vassiliadou and the author obtained in [OV2] and [R4] a pretty complete description of the L^2 -cohomology of the $\bar{\partial}$ -operator (in the sense of distributions) at isolated singularities.

In this setting, we understand the class of objects with which we deal very well (just L^2 -forms), but the disadvantage is a different one. Whereas the $\bar{\partial}$ -equation is locally solvable for $\bar{\partial}$ -closed $(0, q)$ -forms in the category of \mathcal{A} -sheaves by the Koppelman formulas in [AS], there are local obstructions to solving the $\bar{\partial}$ -equation in the L^2 -sense at singular points (see e.g. [FOV], [OV2], [R4]). So, there can be no L^2 - $\bar{\partial}$ -resolution for the structure sheaf in general.

The starting point of the present paper was the idea that the $\bar{\partial}$ -operator in the L^2 -sense may behave very well on spaces with canonical singularities. There are several results which substantiate this idea. First, holomorphic differential forms are reflexive on such spaces (see [GKKP]). It follows that holomorphic differential forms on the regular part of the variety are in L^2 . Second, it is known that the $\bar{\partial}$ -equation is solvable in the L^2 -sense for $(0, q)$ -forms on affine cones over projective varieties of low degree¹ (see [R3] and [OV2], Section 7, Example 3). Third, there is a long list of related positive results on the $\bar{\partial}$ -equation at rational double points² (see [AZ1], [AZ2], [AS], and also [R3], [LR] for the A_1 -singularity).

The underlying idea is that canonical singularities are rational (see e.g. [K], Theorem 11.1), i.e., we expect that the singularities do not contribute to the local cohomology in a certain sense. Pursuing this idea, it turned out that there is a notion of L^2 - $\bar{\partial}$ -cohomology for $(0, q)$ -forms which can be described completely in terms of a resolution of singularities (see Theorem 1.1 below). A singular point is rational if and only if the L^2 - $\bar{\partial}$ -complex is exact in this point. If the underlying space has rational singularities, then we obtain an L^2 - $\bar{\partial}$ -resolution of the structure sheaf, i.e., a resolution of the structure sheaf in terms of a well-known class of differential forms. This gives even more hope for the idea that spaces with canonical singularities may allow for a comprehensive L^2 -theory for the $\bar{\partial}$ -operator.

One of our main tools is a version of topological L^2 -Serre duality for singular complex spaces with arbitrary singularities. In order to establish and apply this kind of Serre duality, we have to overcome several difficulties which do not appear in the smooth setting. We have to establish some new duality relations between different closed L^2 -extensions of the $\bar{\partial}$ -operator at arbitrary singularities, and to provide new Hausdorffness results for some cohomology groups on singular spaces.

¹Such varieties are canonical.

² Rational double points are precisely the possible canonical singularities in dimension two.

To explain our results more precisely, let us introduce some notation (see Section 2.1 for the details). Let X be a Hermitian complex space³ of pure dimension n and $F \rightarrow X$ a Hermitian holomorphic line bundle. We denote by $\mathcal{L}^{p,q}(F)$ the sheaf of germs of F -valued (p, q) -forms on the regular part of X which are square-integrable on $K \setminus \text{Sing } X$ for compact sets K in their domain of definition.⁴

Due to the incompleteness of the metric on $X \setminus \text{Sing } X$, there are different reasonable definitions of the $\bar{\partial}$ -operator on $\mathcal{L}^{p,q}(F)$ -forms. To be more precise, let $\bar{\partial}_{cpt}$ be the $\bar{\partial}$ -operator on smooth forms with support away from the singular set $\text{Sing } X$. Then $\bar{\partial}_{cpt}$ can be considered as a densely defined operator $\mathcal{L}^{p,q}(F) \rightarrow \mathcal{L}^{p,q+1}(F)$. One can now consider various closed extensions of this operator. The two most important are the maximal closed extension, i.e., the $\bar{\partial}$ -operator in the sense of distributions which we denote by $\bar{\partial}_w$, and the minimal closed extension, i.e., the closure of the graph of $\bar{\partial}_{cpt}$ which we denote by $\bar{\partial}_s$. Let $\mathcal{C}^{p,q}(F)$ be the domain of definition of $\bar{\partial}_w$ which is a subsheaf of $\mathcal{L}^{p,q}(F)$, and $\mathcal{F}^{p,q}(F)$ the domain of definition of $\bar{\partial}_s$ which in turn is a subsheaf of $\mathcal{C}^{p,q}(F)$. We obtain complexes of fine sheaves

$$\mathcal{C}^{p,0}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}^{p,1}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}^{p,2}(F) \xrightarrow{\bar{\partial}_w} \dots \quad (1)$$

and

$$\mathcal{F}^{p,0}(F) \xrightarrow{\bar{\partial}_s} \mathcal{F}^{p,1}(F) \xrightarrow{\bar{\partial}_s} \mathcal{F}^{p,2}(F) \xrightarrow{\bar{\partial}_s} \dots \quad (2)$$

If F is just the trivial line bundle, then $\mathcal{K}_X := \ker \bar{\partial}_w \subset \mathcal{C}^{n,0}$ is the canonical sheaf of Grauert–Riemenschneider and $\mathcal{K}_X^s := \ker \bar{\partial}_s \subset \mathcal{F}^{n,0}$ is the sheaf of holomorphic n -forms with a certain boundary condition that was introduced in [R4]. We will see below that $\hat{\mathcal{O}}_X = \ker \bar{\partial}_s \subset \mathcal{F}^{0,0}$ for the sheaf of weakly holomorphic functions $\hat{\mathcal{O}}_X$.

It is clear that (1) and (2) are exact in regular points of X . Exactness in singular points is equivalent to the difficult problem of solving $\bar{\partial}$ -equations locally in the L^2 -sense at singularities, which is not possible in general (see e.g. [FOV], [OV1], [OV2], [R2], [R3], [R4]). However, it is known that (1) is exact for $p = n$, and that (2) is exact for $p = n$ if X has only isolated singularities ([PS] and [R4]; see Section 2.2). In these cases, the complexes (1) and (2) are fine resolutions of the canonical sheaves \mathcal{K}_X and \mathcal{K}_X^s , respectively.

For an open set $\Omega \subset X$, we denote by $H_{w,loc}^{p,q}(\Omega, F)$ the cohomology of the complex (1), and by $H_{w,cpt}^{p,q}(\Omega, F)$ the cohomology of (1) with compact support. Analogously, let $H_{s,loc}^{p,q}(\Omega, F)$ and $H_{s,cpt}^{p,q}(\Omega, F)$ be the cohomology groups of (2). These L^2 -cohomology groups inherit the structure of topological vector spaces, which are locally convex Hausdorff spaces if the corresponding $\bar{\partial}$ -operators have closed range.⁵

We can now formulate the main results of the present paper.

Theorem 1.1. *Let X be a Hermitian complex space, $\pi : M \rightarrow X$ a resolution of singularities and $\Omega \subset X$ a holomorphically convex domain. Then push-forward of forms induces for all $q \geq 0$ a natural topological isomorphism*

$$H^q(\pi^{-1}(\Omega), \mathcal{O}_M) \xrightarrow{\cong} H_{s,loc}^{0,q}(\Omega). \quad (3)$$

³A Hermitian complex space (X, g) is a reduced complex space X with a metric g on the regular part such that the following holds: If $x \in X$ is an arbitrary point there exists a neighborhood $U = U(x)$ and a biholomorphic embedding of U into a domain G in \mathbb{C}^N and an ordinary smooth Hermitian metric in G whose restriction to U is $g|_U$.

⁴This is what we mean by square-integrable up to the singular set.

⁵Note that different Hermitian metrics lead to $\bar{\partial}$ -complexes which are equivalent on relatively compact subsets. So, one can put any Hermitian metric on X in many of the results below.

From that we obtain for the local $L^2\text{-}\bar{\partial}_s$ -cohomology immediately:

Theorem 1.2. *Let X be a Hermitian complex space, $\pi : M \rightarrow X$ a resolution of singularities and $q \geq 0$. Then:*

$$(\mathcal{H}^q(\mathcal{F}^{0,*}))_x \cong (R^q\pi_*\mathcal{O}_M)_x \quad \forall x \in X. \quad (4)$$

Particularly, for $q = 0$,

$$\ker \bar{\partial}_s^{0,0} = \mathcal{H}^0(\mathcal{F}^{0,*}) = \hat{\mathcal{O}}_X, \quad (5)$$

where $\hat{\mathcal{O}}_X$ denotes the sheaf of germs of weakly holomorphic functions.

It follows that $x \in X$ is a normal point exactly if $(\ker \bar{\partial}_s^{0,0})_x = \mathcal{O}_{X,x}$. As $x \in X$ is a rational point if it is normal and $(R^q\pi_*\mathcal{O}_M)_x = 0$ for $q > 0$, we get also:

Theorem 1.3. *Let X be a Hermitian complex space. Then the $L^2\text{-}\bar{\partial}$ -complex*

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{F}^{0,0} \xrightarrow{\bar{\partial}_s} \mathcal{F}^{0,1} \xrightarrow{\bar{\partial}_s} \mathcal{F}^{0,2} \xrightarrow{\bar{\partial}_s} \mathcal{F}^{0,3} \xrightarrow{\bar{\partial}_s} \dots \quad (6)$$

is exact in a point $x \in X$ if and only if x is a rational point.

Hence, if X has only rational singularities, then (6) is a fine resolution of the structure sheaf \mathcal{O}_X .

If X has only rational singularities, then Theorem 1.3 yields directly further finiteness and vanishing results, e.g. if X is q -convex or q -complete. An essential tool in the proof of Theorem 1.1 is the following new version of topological L^2 -Serre duality on singular complex spaces with arbitrary singularities:

Theorem 1.4 (Serre duality). *Let X be a Hermitian complex space of pure dimension n , $\Omega \subset X$ an open set, $F \rightarrow X$ a Hermitian holomorphic line bundle, and let $0 \leq p, q \leq n$. If $H_{w,loc}^{p,q}(\Omega, F)$ and $H_{w,loc}^{p,q+1}(\Omega, F)$ are Hausdorff, then the mapping*

$$\mathcal{L}^{p,q}(\Omega, F) \times \mathcal{L}_{cpt}^{n-p,n-q}(\Omega, F^*) \rightarrow \mathbb{C} \quad , \quad (\eta, \omega) \mapsto \int_{\Omega^*} \eta \wedge \omega,$$

induces a non-degenerate pairing of topological vector spaces

$$H_{w,loc}^{p,q}(\Omega, F) \times H_{s,cpt}^{n-p,n-q}(\Omega, F^*) \rightarrow \mathbb{C}$$

such that $H_{s,cpt}^{n-p,n-q}(\Omega, F^*)$ is the topological dual of $H_{w,loc}^{p,q}(\Omega, F)$ and vice versa.

The same statement holds with the indices $\{s, w\}$ in place of $\{w, s\}$. Then there is a non-degenerate pairing

$$H_{s,loc}^{p,q}(\Omega, F) \times H_{w,cpt}^{n-p,n-q}(\Omega, F^*) \rightarrow \mathbb{C}.$$

If the topological vector spaces $H_{w/s,loc}^{p,q}(\Omega, F)$, $H_{w/s,loc}^{p,q+1}(\Omega, F)$ are non-Hausdorff, then the statement of Theorem 1.4 holds at least for the separated cohomology groups $\overline{H}_{w/s} = \ker \bar{\partial}_{w/s} / \overline{\text{Im } \bar{\partial}_{w/s}}$ (Theorem 3.5).⁶ Though Theorem 1.4 looks at first glance like a standard version of Serre duality, there are two essential difficulties in its proof which do usually not appear. First, the $\bar{\partial}$ -operators under consideration are just closed densely defined operators in the Fréchet spaces $\mathcal{L}^{p,q}(\Omega, F)$ and the (LF) -spaces $\mathcal{L}_{cpt}^{n-p,n-q}(\Omega, F^*)$. Second, we have to show that the operators $\bar{\partial}_w$ and $\bar{\partial}_s$ are topologically dual at arbitrary singularities. This point is the main technical difficulty in this paper.

⁶The notation w/s refers either to the index w or the index s in the whole statement.

Note that $H_{w/s,loc}^{p,q}(\Omega, F)$ is Hausdorff if and only if $\bar{\partial}_{w/s}$ has closed range in $\mathcal{L}^{p,q}(\Omega, F)$, and to decide whether this is the case is usually as difficult as solving the corresponding $\bar{\partial}$ -equation. Using local L^2 - $\bar{\partial}$ -solution results for singular spaces and the theory of Fréchet sheaves, we will show at least:

Theorem 1.5. *Let X be a Hermitian complex space of pure dimension n , $F \rightarrow X$ a Hermitian holomorphic line bundle, and let $0 \leq p, q \leq n$. Let $\Omega \subset X$ be a holomorphically convex open subset. Then the topological vector spaces*

$$H_{w,loc}^{n,q}(\Omega, F) \quad , \quad H_{w,cpt}^{n,q}(\Omega, F) \quad , \quad H_{s,cpt}^{0,n-q}(\Omega, F^*) \quad , \quad H_{s,loc}^{0,n-q}(\Omega, F^*)$$

are Hausdorff for all $0 \leq q \leq n$.

If X has only isolated singularities, then the topological vector spaces

$$H_{s,loc}^{n,q}(\Omega, F) \quad , \quad H_{w,cpt}^{0,n-q}(\Omega, F^*)$$

are Hausdorff for all $0 \leq q \leq n$, too.

If X has only homogeneous (conical) isolated singularities, then the topological vector spaces

$$H_{s,cpt}^{n,q}(\Omega, F) \quad , \quad H_{w,loc}^{0,n-q}(\Omega, F^*)$$

are Hausdorff for all $0 \leq q \leq n$, too.

A main point in the proof of Theorem 1.5 is to show that the canonical Fréchet sheaf structure of compact convergence on the coherent analytic canonical sheaves \mathcal{K}_X and \mathcal{K}_X^s , respectively, coincides with the Fréchet sheaf structure of L^2 -convergence on compact subsets (Theorem 3.6). This allows then to show also the topological equivalence of Čech cohomology and L^2 -cohomology (Theorem 3.8).

As a direct application of our Serre duality, Theorem 1.4, we deduce again by use of local $\bar{\partial}$ -solution results for (n, q) -forms and the equivalence of Fréchet structures, Theorem 3.8, another main result:

Theorem 1.6. *Let X be a Hermitian complex space of pure dimension n , $F \rightarrow X$ a Hermitian holomorphic line bundle and $\Omega \subset X$ a cohomologically q -complete open subset, $q \geq 1$. Then*

$$H_{w,loc}^{n,r}(\Omega, F) = H_{s,cpt}^{0,n-r}(\Omega, F^*) = 0 \quad \text{for all } r \geq q.$$

If X has only isolated singularities, then also

$$H_{s,loc}^{n,r}(\Omega, F) = H_{w,cpt}^{0,n-r}(\Omega, F^*) = 0 \quad \text{for all } r \geq q.$$

Note that Ω is cohomologically q -complete if it is q -complete by the Andreotti-Grauert vanishing theorem [AG]. So, Theorem 1.6 allows to solve the $\bar{\partial}_s$ -equation with compact support for $(0, n-q)$ -forms on q -complete spaces, which is of particular interest for 1-complete spaces, i.e., Stein spaces.

It is thus interesting to understand the $\bar{\partial}_s$ -operator better. In contrast to the $\bar{\partial}$ -operator in the sense of distributions (the $\bar{\partial}_w$ -operator), $\bar{\partial}_s$ comes with a certain kind of boundary (respectively growth) condition at the singular set. It follows by an argument of N. Sibony from [S1] that locally bounded forms in the domain of the $\bar{\partial}_w$ -operator are also in the domain of the $\bar{\partial}_s$ -operator (see Theorem 4.1). This is a direct way to see that weakly holomorphic functions are $\bar{\partial}_s$ -closed (cf. Theorem 1.2).

To exemplify the use of $\bar{\partial}$ -equations on singular spaces, we give as an application a short proof of the Hartogs' extension theorem in its most general form:

Theorem 1.7. *Let X be a connected normal complex space of dimension $n \geq 2$ which is cohomologically $(n - 1)$ -complete. Furthermore, let D be a domain in X and $K \subset D$ a compact subset such that $D \setminus K$ is connected. Then each holomorphic function $f \in \mathcal{O}(D \setminus K)$ has a unique holomorphic extension to the whole set D .*

In this generality, Hartogs' extension theorem is due to Merker and Porten [MP], who proved also extension of meromorphic functions. Merker and Porten gave an involved geometrical proof by using a finite number of parameterized families of holomorphic discs and Morse-theoretical tools for the global topological control of monodromy, but no $\bar{\partial}$ -theory. Shortly after that, Coltoiu and Ruppenthal obtained a $\bar{\partial}$ -theoretical proof by the Ehrenpreis- $\bar{\partial}$ -technique (see [CR]). This approach involves Hironaka's deep results on resolution of singularities. It is thus a natural question to ask whether the extension can be achieved by working on the original singular space only. In the present paper, we obtain a very short proof of Theorem 1.7 by the Ehrenpreis- $\bar{\partial}$ -technique without needing a resolution of singularities. We just use the $\bar{\partial}$ -vanishing $H_{s,cpt}^{0,1}(X) = 0$ for an $(n - 1)$ -complete space X and the fact that bounded $\bar{\partial}$ -closed forms are in the kernel of $\bar{\partial}_s$ (see Section 4.3).

Let us also mention that one can show $H_{L^\infty,cpt}^{0,1}(X) = 0$ for an $(n - 1)$ -complete space X by using a resolution of singularities and Takegoshi's vanishing theorem (see Theorem 4.3).

Let us point out also the following interesting fact. Let X be a Gorenstein space with canonical singularities. By exactness of (6) and exactness of (1) for $p = n$, the non-degenerate L^2 -Serre duality pairing

$$H_{s,loc}^{0,q}(\Omega) \times H_{w,cpt}^{n,n-q}(\Omega) \rightarrow \mathbb{C}, \quad ([\eta], [\omega]) \mapsto \int_{\Omega^*} \eta \wedge \omega,$$

is for $0 \leq q \leq n$ then an explicit realization of Grothendieck duality after Ramis-Ruget [RR],

$$H^q(\Omega, \mathcal{O}_X) \cong H_{cpt}^{n-q}(\Omega, \omega_X),$$

given the cohomology groups under consideration are Hausdorff. Here, ω_X denotes the Grothendieck dualizing sheaf which coincides with the Grauert-Riemenschneider canonical sheaf \mathcal{K}_X as X has canonical Gorenstein singularities.

The present paper is organized as follows. In Section 2 we provide the necessary preliminaries: the $\bar{\partial}_w$ - and the $\bar{\partial}_s$ -complex, some L^2 - $\bar{\partial}$ -solution results and their direct consequences, L^2 - $\bar{\partial}$ -Hilbert space theory, topological preliminaries, Fréchet sheaves. In Section 3, we prove Serre duality, Theorem 1.4, and study the equivalence of the topology of compact convergence and L^2 -topology which leads to Theorem 1.5 and Theorem 1.6. Section 4 is then devoted to the study of the $\bar{\partial}_s$ -operator and Hartogs' extension theorem. Finally, we prove Theorem 1.1 and its consequences (Theorem 1.2 and Theorem 1.3) in the last section.

2. PRELIMINARIES

2.1. Two $\bar{\partial}$ -complexes on singular spaces. Let us recall some of the essential constructions from [R4].

Let X be a (singular) Hermitian complex space of pure dimension n . For any subset $S \subset X$, we use the notation S^* for $S \setminus \text{Sing } X$. Let $F \rightarrow X^*$ be a Hermitian holomorphic line bundle and $U \subset X$ an open subset. On a singular space, it is

fruitful to consider forms that are square-integrable up to the singular set. Hence, we use the following concept of locally square-integrable forms:

$$L_{loc}^{p,q}(U, F) := \{f \in L_{loc}^{p,q}(U^*, F) : f|_K \in L^{p,q}(K^*, F) \ \forall K \subset\subset U\}.$$

It is easy to check that the presheaves given as

$$\mathcal{L}^{p,q}(U, F) := L_{loc}^{p,q}(U, F)$$

are already sheaves $\mathcal{L}^{p,q}(F) \rightarrow X$. On $L_{loc}^{p,q}(U, F)$, we denote by

$$\bar{\partial}_w(U) : L_{loc}^{p,q}(U, F) \rightarrow L_{loc}^{p,q+1}(U, F)$$

the $\bar{\partial}$ -operator in the sense of distributions on $U^* = U \setminus \text{Sing } X$ which is closed and densely defined. When there is no danger of confusion, we will simply write $\bar{\partial}_w$ for $\bar{\partial}_w(U)$. The subscript refers to $\bar{\partial}_w$ as an operator in a weak sense. Since $\bar{\partial}_w$ is a local operator, i.e. $\bar{\partial}_w(U)|_V = \bar{\partial}_w(V)$ for open sets $V \subset U$, we can define the presheaves of germs of forms in the domain of $\bar{\partial}_w$,

$$\mathcal{C}^{p,q}(F) := \mathcal{L}^{p,q}(F) \cap \bar{\partial}_w^{-1} \mathcal{L}^{p,q+1}(F),$$

given by

$$\mathcal{C}^{p,q}(U, F) = \mathcal{L}^{p,q}(U, F) \cap \text{Dom } \bar{\partial}_w(U).$$

These are actually already sheaves because the following is also clear: If $U = \bigcup U_\mu$ is a union of open sets, $f_\mu = f|_{U_\mu}$ and

$$f_\mu \in \text{Dom } \bar{\partial}_w(U_\mu),$$

then

$$f \in \text{Dom } \bar{\partial}_w(U) \quad \text{and} \quad (\bar{\partial}_w(U)f)|_{U_\mu} = \bar{\partial}_w(U_\mu)f_\mu.$$

Moreover, it is easy to see that the sheaves $\mathcal{C}^{p,q}(F)$ admit partitions of unity, and so we obtain a complex of fine sheaves

$$\mathcal{C}^{p,0}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}^{p,1}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}^{p,2}(F) \xrightarrow{\bar{\partial}_w} \dots \quad (7)$$

We use simply $\mathcal{C}^{p,q}$ to denote the sheaves of forms with values in the trivial line bundle. We define

$$\mathcal{K}_X(F) := \ker \bar{\partial}_w \subset \mathcal{C}^{n,0}(F). \quad (8)$$

Using a resolution of singularities, one sees that $\mathcal{K}_X := \ker \bar{\partial}_w \subset \mathcal{C}^{n,0}$ is just the canonical sheaf of Grauert and Riemenschneider because the L^2 -property of $(n,0)$ -forms remains invariant under modifications (see [R4], Section 2.2).

The $L^{2,loc}$ -Dolbeault cohomology for forms with values in F with respect to the $\bar{\partial}_w$ -operator on an open set $U \subset X$ is by definition the cohomology of the complex (7) which is denoted by $H^q(\Gamma(U, \mathcal{C}^{p,*}(F)))$. The cohomology with compact support is $H^q(\Gamma_{cpt}(U, \mathcal{C}^{p,*}(F)))$. Note that this is the cohomology of forms with compact support in U , not with compact support in $U^* = U \setminus \text{Sing } X$.

We use also the following notation for the $\bar{\partial}_w$ -cohomology:

Definition 2.1. *For an open set $\Omega \subset X$ and a Hermitian holomorphic line bundle $F \rightarrow X$, let*

$$\begin{aligned} H_{w,loc}^{p,q}(\Omega, F) &:= H^q(\Gamma(\Omega, \mathcal{C}^{p,*}(F))), \\ H_{w,cpt}^{p,q}(\Omega, F) &:= H^q(\Gamma_{cpt}(\Omega, \mathcal{C}^{p,*}(F))). \end{aligned}$$

Secondly, we introduce a suitable local realization of a minimal version of the $\bar{\partial}$ -operator. This is the $\bar{\partial}$ -operator with a certain boundary condition at the singular set $\text{Sing } X$ of X . Let

$$\bar{\partial}_s(U) : L_{loc}^{p,q}(U, F) \rightarrow L_{loc}^{p,q+1}(U, F)$$

be defined as follows.⁷ We say that $f \in \text{Dom } \bar{\partial}_w$ is in the domain of $\bar{\partial}_s$ if there exists a sequence of forms $\{f_j\}_j \subset \text{Dom } \bar{\partial}_w \subset L_{loc}^{p,q}(U, F)$ with essential support away from the singular set,

$$\text{supp } f_j \cap \text{Sing } X = \emptyset,$$

such that

$$f_j \rightarrow f \quad \text{in} \quad L^{p,q}(K^*, F), \quad (9)$$

$$\bar{\partial}_w f_j \rightarrow \bar{\partial}_w f \quad \text{in} \quad L^{p,q+1}(K^*, F) \quad (10)$$

for each compact subset $K \subset\subset U$. The subscript refers to $\bar{\partial}_s$ as an extension in a strong sense. Note that we can assume without loss of generality (by use of cut-off functions and smoothing with Dirac sequences) that the forms f_j are smooth with compact support in $U^* = U - \text{Sing } X$. This is the equivalent definition that we used in [R4] where we denoted the operator by $\bar{\partial}_{s,loc}$.

It is now clear that $\bar{\partial}_s(U)|_V = \bar{\partial}_s(V)$ for open sets $V \subset U$, and we can define the presheaves of germs of forms in the domain of $\bar{\partial}_s$,

$$\mathcal{F}^{p,q}(F) := \mathcal{L}^{p,q}(F) \cap \bar{\partial}_s^{-1} \mathcal{L}^{p,q+1}(F),$$

given by

$$\mathcal{F}^{p,q}(U, F) = \mathcal{L}^{p,q}(U, F) \cap \text{Dom } \bar{\partial}_s(U).$$

Here, we shall check a bit more carefully that these are already sheaves: Let $U = \bigcup U_\mu$ be a union of open sets, $f \in L_{loc}^{p,q}(U, F)$ and $f_\mu = f|_{U_\mu} \in \text{Dom } \bar{\partial}_s(U_\mu)$ for all μ . We claim that $f \in \text{Dom } \bar{\partial}_s(U)$. To see this, we can assume (by taking a refinement if necessary) that the open cover $\mathcal{U} := \{U_\mu\}_\mu$ is locally finite, and choose a partition of unity $\{\varphi_\mu\}_\mu$ for \mathcal{U} . On U_μ choose a sequence $\{f_j^\mu\}_j \subset L_{loc}^{p,q}(U_\mu, F)$ as in (9), (10), and consider $f_j := \sum_\mu \varphi_\mu f_j^\mu$. It is clear that $\{f_j\}_j \subset L_{loc}^{p,q}(U, F)$. If $K \subset\subset U$ is compact, then $K \cap \text{supp } \varphi_\mu$ is a compact subset of U_μ for each μ , so that $\{f_j^\mu\}_j$ and $\{\bar{\partial} f_j^\mu\}_j$ converge in the L^2 -sense to f_μ resp. $\bar{\partial}_w f_\mu$ on $K \cap \text{supp } \varphi_\mu$. But then $\{f_j\}_j$ and $\{\bar{\partial} f_j\}_j$ converge in the L^2 -sense to f resp. $\bar{\partial}_w f$ on K (recall that the cover is locally finite) and that is what we had to show.

As for $\mathcal{C}^{p,q}(F)$, it is clear that the sheaves $\mathcal{F}^{p,q}(F)$ are fine, and we obtain a complex of fine sheaves

$$\mathcal{F}^{p,0}(F) \xrightarrow{\bar{\partial}_s} \mathcal{F}^{p,1}(F) \xrightarrow{\bar{\partial}_s} \mathcal{F}^{p,2}(F) \xrightarrow{\bar{\partial}_s} \dots \quad (11)$$

Again, we use simply $\mathcal{F}^{p,q}$ to denote the sheaves of forms with values in the trivial line bundle. We define the canonical sheaf of holomorphic n -forms with a Dirichlet boundary condition:

$$\mathcal{K}_X^s(F) := \ker \bar{\partial}_s \subset \mathcal{F}^{n,0}(F).$$

Definition 2.2. For $\Omega \subset X$ open, we use the notation:

$$\begin{aligned} H_{s,loc}^{p,q}(\Omega, F) &:= H^q(\Gamma(\Omega, \mathcal{F}^{p,*}(F))), \\ H_{s,cpt}^{p,q}(\Omega, F) &:= H^q(\Gamma_{cpt}(\Omega, \mathcal{F}^{p,*}(F))). \end{aligned}$$

⁷Again, we write simply $\bar{\partial}_s$ for $\bar{\partial}_s(U)$ if there is no danger of confusion.

2.2. Local L^2 -solvability for (n, q) -forms. It is clearly interesting to study whether the sequences (7) and (11) are exact, which is well-known to be the case in regular points of X where the $\bar{\partial}_w$ - and the $\bar{\partial}_s$ -operator coincide. In singular points, the situation is quite complicated for forms of arbitrary degree and not completely understood. However, the $\bar{\partial}_w$ -equation is locally solvable in the L^2 -sense at arbitrary singularities for forms of degree (n, q) , $q > 0$ ([PS], Proposition 2.1), and for forms of degree (p, q) , $p + q > n$, at isolated singularities ([FOV], Theorem 1.2). We may restrict ourselves here to the case of (n, q) -forms and have (see [R4], Theorem 3.1):

Theorem 2.3. *Let X be a Hermitian complex space of pure dimension n , and $F \rightarrow X^* = X \setminus \text{Sing } X$ a Hermitian holomorphic line bundle which is locally semi-positive with respect to X , i.e. for each point $x \in X$ there is a neighborhood $U_x \subset X$ such that F is semi-positive on $U_x^* = U_x \setminus \text{Sing } X$. Then the complex*

$$0 \rightarrow \mathcal{K}_X(F) \longrightarrow \mathcal{C}^{n,0}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}^{n,1}(F) \xrightarrow{\bar{\partial}_w} \mathcal{C}^{n,2}(F) \xrightarrow{\bar{\partial}_w} \dots \quad (12)$$

is exact, i.e. it is a fine resolution of $\mathcal{K}_X(F)$. For an open set $U \subset X$, it follows that

$$H^q(U, \mathcal{K}_X(F)) \cong H_{w,loc}^{n,q}(U, F) \quad , \quad H_{cpt}^q(U, \mathcal{K}_X(F)) \cong H_{w,cpt}^{n,q}(U, F).$$

Note that the assumption on F is trivially fulfilled if F extends to a holomorphic line bundle over X . For the case of the trivial line bundle, $F = X \times \mathbb{C}$, Theorem 2.3 is due to Pardon-Stern ([PS], Proposition 2.1).

Concerning the $\bar{\partial}_s$ -equation, local L^2 -solvability for forms of degree (n, q) is known to hold on spaces with isolated singularities (see [R4], Lemma 5.4 and Lemma 6.3), but the problem is open at arbitrary singularities.

So, let X have only isolated singularities. Then the $\bar{\partial}_s$ -equation is locally exact on (n, q) -forms for $1 \leq q \leq n-1$ by [R4], Lemma 5.4, and for $q \geq 2$ by [R4], Lemma 6.3. Both statements were deduced from the results of Fornæss, Øvrelid and Vassiliadou [FOV]. The case of $\dim X = n = 1$ is treated in [RS]. Hence:

Theorem 2.4. *Let X be a Hermitian complex space of pure dimension n with only isolated singularities. Then*

$$0 \rightarrow \mathcal{K}_X^s \hookrightarrow \mathcal{F}^{n,0} \xrightarrow{\bar{\partial}_s} \mathcal{F}^{n,1} \xrightarrow{\bar{\partial}_s} \mathcal{F}^{n,2} \xrightarrow{\bar{\partial}_s} \dots \longrightarrow \mathcal{F}^{n,n} \rightarrow 0 \quad (13)$$

is a fine resolution. For an open set $U \subset X$, it follows that

$$H^q(U, \mathcal{K}_X^s) \cong H_{s,loc}^{n,q}(U) \quad , \quad H_{cpt}^q(U, \mathcal{K}_X^s) \cong H_{s,cpt}^{n,q}(U).$$

2.3. Resolution of singularities and Takegoshi's vanishing theorem. We need to recall some more material from [R4], Section 2.2. For X as above and $F \rightarrow X$ a Hermitian holomorphic line bundle, let $\pi : M \rightarrow X$ be a resolution of singularities (which exists due to Hironaka), and give M an arbitrary positive definite Hermitian metric σ . Then we denote analogously to (7) by

$$0 \rightarrow \mathcal{K}_M(\pi^*F) \longrightarrow \mathcal{C}_\sigma^{n,0}(\pi^*F) \xrightarrow{\bar{\partial}_w} \mathcal{C}_\sigma^{n,1}(\pi^*F) \xrightarrow{\bar{\partial}_w} \mathcal{C}_\sigma^{n,2}(\pi^*F) \xrightarrow{\bar{\partial}_w} \dots \quad (14)$$

the fine $L_\sigma^{2,loc}$ -resolution of the canonical sheaf $\mathcal{K}_M(\pi^*F)$ on M with values in π^*F (it is well known that (14) is exact). As \mathcal{K}_X is the Grauert-Riemenschneider canonical sheaf on X , we have (see [R4], Theorem 2.1):

$$\mathcal{K}_X(F) = \pi_* \mathcal{K}_M(\pi^*F). \quad (15)$$

Takegoshi's vanishing theorem [T] yields the vanishing of the higher direct image sheaves:

$$R^q \pi_* \mathcal{K}_M(\pi^* F) = 0, \quad q > 0. \quad (16)$$

Moreover, square-integrable (n, q) -forms remain square-integrable under pull-back by π and this pull-back commutes with the $\bar{\partial}_w$ -operator. The exceptional set of the resolution $\pi : M \rightarrow X$ does no harm as the $\bar{\partial}$ -equation in the L^2 -sense extends over hypersurfaces. So, π induces a natural mapping of complexes

$$\pi^* : (\mathcal{C}^{n,*}(F), \bar{\partial}_w) \longrightarrow (\pi_* \mathcal{C}_\sigma^{n,*}(\pi^* F), \pi_* \bar{\partial}_w), \quad (17)$$

and by Theorem 2.3 and (16), both complexes in (17) are fine resolutions of $\mathcal{K}_X(F) = \pi_* \mathcal{K}_M(\pi^* F)$. As π^* commutes with the $\bar{\partial}_w$ -operator, it induces isomorphisms

$$H_{w,loc}^{n,q}(\Omega, F) \xrightarrow{\cong} H_{w,loc}^{n,q}(\pi^{-1}(\Omega), \pi^* F) \cong H^q(\pi^{-1}(\Omega), \mathcal{K}_M(\pi^* F)), \quad (18)$$

$$H_{w,cpt}^{n,q}(\Omega, F) \xrightarrow{\cong} H_{w,cpt}^{n,q}(\pi^{-1}(\Omega), \pi^* F) \cong H_{cpt}^q(\pi^{-1}(\Omega), \mathcal{K}_M(\pi^* F)) \quad (19)$$

for any open set $\Omega \subset X$ and $0 \leq q \leq n$ (see [R4], Theorem 2.1). We will see later (Section 3.7) that the algebraic isomorphisms in (18), (19) are also topological isomorphisms.

For the rest of this section, assume that X has only homogeneous isolated singularities. In this situation, we have $\mathcal{K}_X^s \cong \mathcal{K}_X$ by [R4], Theorem 1.10 (with $D = \emptyset$), because homogeneous isolated singularities can be resolved by a single blow-up.

As the $\bar{\partial}_s$ -operator is stronger than the $\bar{\partial}_w$ -operator, π induces also a natural mapping of complexes

$$\pi^* : (\mathcal{F}^{n,*}(F), \bar{\partial}_s) \longrightarrow (\pi_* \mathcal{C}_\sigma^{n,*}(\pi^* F), \pi_* \bar{\partial}_w). \quad (20)$$

By Theorem 2.4 and (16), both complexes in (20) are fine resolutions of $\mathcal{K}_X^s(F) = \pi_* \mathcal{K}_M(\pi^* F)$ and so π^* induces also isomorphisms

$$H_{s,loc}^{n,q}(\Omega, F) \xrightarrow{\cong} H_{w,loc}^{n,q}(\pi^{-1}(\Omega), \pi^* F) \cong H^q(\pi^{-1}(\Omega), \mathcal{K}_M(\pi^* F)), \quad (21)$$

$$H_{s,cpt}^{n,q}(\Omega, F) \xrightarrow{\cong} H_{w,cpt}^{n,q}(\pi^{-1}(\Omega), \pi^* F) \cong H_{cpt}^q(\pi^{-1}(\Omega), \mathcal{K}_M(\pi^* F)) \quad (22)$$

for any open set $\Omega \subset X$ and $0 \leq q \leq n$. Again, we will see in Section 3.7 that (21), (22) are also topological isomorphisms.

2.4. Metrizable topology of $L_{loc}^{p,q}(\Omega, F)$. Let $\Omega \subset X$ be an open subset. We give $L_{loc}^{p,q}(\Omega, F)$ the structure of a Fréchet space with the topology of L^2 -convergence on compact subsets. This topology is obtained as follows. Let $K_1 \subset K_2 \subset K_3 \subset \dots \subset \Omega$ be a compact exhaustion of Ω , and define the separating family of seminorms

$$p_j(\eta) := \left(\int_{K_j^*} |\eta|_F^2 dV_X \right)^{1/2} \quad (23)$$

for $\eta \in L_{loc}^{p,q}(\Omega, F)$ and $j = 1, 2, \dots$, where dV_X is the volume form on X^* induced by the Hermitian metric of the Hermitian space X . $L_{loc}^{p,q}(\Omega, F)$ is then a Fréchet space with the metric

$$d(\eta, \omega) := \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(\eta - \omega)}{1 + p_j(\eta - \omega)}, \quad \eta, \omega \in L_{loc}^{p,q}(\Omega, F)$$

(compare e.g. [R1], Theorem 1.37, Remark 1.38 and Example 1.44). The induced topology is also called the topology of compact L^2 -convergence. It is not hard to see that this topology does not depend on the compact exhaustion.

2.5. Dual space of $L_{loc}^{p,q}(\Omega, F)$. Let us consider the vector space of compactly supported L^2 -forms with values in the (Hermitian) dual bundle F^* ,

$$L_{cpt}^{r,s}(\Omega, F^*) := \{\eta \in L_{loc}^{r,s}(\Omega, F^*) : \text{supp } \eta \subset\subset \Omega\},$$

which inherits a metric as a subspace of $L_{loc}^{r,s}(\Omega, F^*)$. Unfortunately, this metric is not complete, i.e. $L_{cpt}^{r,s}(\Omega, F^*)$ is not a closed subspace of $L_{loc}^{r,s}(\Omega, F^*)$.

However, analogously to the Schwartz topology on spaces of test-forms, we can give $L_{cpt}^{r,s}(\Omega, F^*)$ the structure of a complete locally convex topological vector space as follows. For a fixed compact set $K \subset \Omega$, let

$$\mathcal{D}_K^{r,s} := L_K^{r,s}(\Omega, F^*) := \{\eta \in L_{loc}^{r,s}(\Omega, F^*) : \text{supp } \eta \subset K\},$$

carrying the induced Fréchet space structure, and let τ_K be the topology on $\mathcal{D}_K^{r,s}$. Following [R1], Definition 6.3, let β be the collection of all convex balanced sets $W \subset L_{cpt}^{r,s}(\Omega, F^*)$ such that $\mathcal{D}_K^{r,s} \cap W \in \tau_K$ for every compact set $K \subset \Omega$, and define the topology τ on $L_{cpt}^{r,s}(\Omega, F^*)$ as the collection of all unions of sets of the form $\phi + W$ with $\phi \in L_{cpt}^{r,s}(\Omega, F^*)$ and $W \in \beta$.

This definition means nothing else but that $L_{cpt}^{r,s}(\Omega, F^*)$ is the inductive limit of the Fréchet spaces $\mathcal{D}_K^{r,s}$. So, $L_{cpt}^{r,s}(\Omega, F^*)$ is an (LF)-space in the sense of Dieudonné-Schwartz [DS1] (consider e.g. their first example of an (LF)-space).

Theorem 2.5. *τ is a topology, making $L_{cpt}^{r,s}(\Omega, F^*)$ into an (LF)-space. Particularly, $L_{cpt}^{r,s}(\Omega, F^*)$ is a locally convex topological vector space and every Cauchy sequence converges. For a sequence of forms $\{\phi_k\}_{k \geq 1} \subset L_{cpt}^{r,s}(\Omega, F^*)$, we have $\phi_k \rightarrow 0$ in the topology τ exactly if*

- *there exists a compact set $K \subset \Omega$ such that $\text{supp } \phi_k \subset K$ for all $k \geq 1$, and*
- *$\phi_k \rightarrow 0$ in $L_K^{r,s}(\Omega, F^*)$.*

Proof. Is contained in [DS1], Section 3 and Section 4. See also [R1], Theorem 6.4 and Theorem 6.5. \square

We will use later the fact that the open mapping theorem is valid in the category of (LF)-spaces ([DS1], Theorem 1).

We can now show:

Theorem 2.6. *Under the non-degenerate pairing*

$$L_{loc}^{p,q}(\Omega, F) \times L_{cpt}^{n-p, n-q}(\Omega, F^*) \rightarrow \mathbb{C}, \quad (\eta, \omega) \mapsto \int_{\Omega^*} \eta \wedge \omega, \quad (24)$$

$L_{cpt}^{n-p, n-q}(\Omega, F^)$ is the topological dual of $L_{loc}^{p,q}(\Omega, F)$, and, vice versa, $L_{loc}^{p,q}(\Omega, F)$ is the topological dual of $L_{cpt}^{n-p, n-q}(\Omega, F^*)$.*

Proof. It is clear that the pairing (24) is well-defined and non-degenerate.

Let $\omega \in L_{cpt}^{n-p, n-q}(\Omega, F^*)$. Then it is easy to see that ω defines a continuous linear functional on $L_{loc}^{p,q}(\Omega, F)$. Assume conversely that $\Lambda \in (L_{loc}^{p,q}(\Omega, F))'$. Recall that the metric of $L_{loc}^{p,q}(\Omega, F)$ is induced by the separating semi-norms p_j in the sense of [R1], Theorem 1.37 (see (23)). Assign to each p_j and each positive integer n the set

$$V(j, n) := \{\eta : p_j(\eta) < 1/n\}.$$

Then the collection of all finite intersections of the sets $V(j, n)$ is a convex balanced local base for the topology of $L_{loc}^{p,q}(\Omega, F)$ (see [R1], Theorem 1.37). As $p_1(\eta) \leq p_2(\eta) \leq p_3(\eta) \leq \dots$ in our situation, already the collection of the sets $V(j, n)$ is a convex local base for the topology. So, by continuity of Λ , there exists indices j_0 and n_0 such that

$$|\Lambda(\eta)| \leq 1 \quad \text{for all } \eta \in V(j_0, n_0).$$

But then

$$|\Lambda(\eta)| \leq n_0 p_{j_0}(\eta) = n_0 \left(\int_{K_{j_0}^*} |\eta|_F^2 dV_X \right)^{1/2} \quad \forall \eta \in L_{loc}^{p,q}(\Omega, F).$$

Thus Λ must have compact support on $K := K_{j_0}$, i.e. $\Lambda(\eta) = \Lambda(\eta|_K)$.

On the other hand, by trivial extension, we have a continuous inclusion

$$L^{p,q}(K^*, F) \subset L_{loc}^{p,q}(\Omega, F),$$

where $L^{p,q}(K^*, F)$ carries the usual L^2 -Hilbert space topology (a sequence converging in $L^{p,q}(K^*, F)$ is, after trivial extension, also converging in $L_{loc}^{p,q}(\Omega, F)$). So, Λ is also a continuous linear functional on $L^{p,q}(K^*, F)$ and thus represented on K^* by an L^2 -form ω_K . But as Λ has support in K , this means that Λ is represented by ω_K on all of Ω by extending ω_K trivially to Ω^* .

For the second statement of the theorem, let $\eta \in L_{loc}^{p,q}(\Omega, F)$. Then it is easy to see that η defines a continuous linear functional on $L_{cpt}^{n-p, n-q}(\Omega, F^*)$. Assume conversely that $\Lambda \in (L_{cpt}^{n-p, n-q}(\Omega, F^*))'$. But

$$L^{n-p, n-q}(V^*, F^*) \cong \mathcal{D}_V^{n-p, n-q} = L_V^{n-p, n-q}(\Omega, F^*) \subset L_{cpt}^{n-p, n-q}(\Omega, F^*)$$

as topological subspaces for any compact subset $V \subset\subset \Omega$. So, Λ is represented by an L^2 -form η_V on any such set V . But the η_V must coincide where their domains intersect, so that Λ is represented by a globally defined form $\eta \in L_{loc}^{p,q}(\Omega, F)$. \square

We remark that $L_{cpt}^{n-p, n-q}(\Omega, F^*)$ is actually the strong topological dual of $L_{loc}^{p,q}(\Omega, F)$, i.e. its topology coincides with the strong dual topology of uniform convergence on bounded subsets of $L_{loc}^{p,q}(\Omega, F)$. To see that, note that a set $E \subset L_{loc}^{p,q}(\Omega, F)$ is bounded exactly if every semi-norm p_j is bounded on E (see [R1], Theorem 1.37 (b)).

Conversely, one can check that $L_{loc}^{p,q}(\Omega, F)$ is actually the strong topological dual of $L_{cpt}^{n-p, n-q}(\Omega, F^*)$. To see that, note that a set $E \subset L_{cpt}^{n-p, n-q}(\Omega, F^*)$ is bounded exactly if there exists a compact set $K \subset \Omega$ such that E is a bounded subset of $L_K^{n-p, n-q}(\Omega, F^*)$ (see [R1], Theorem 6.5 (c)).

We omit the details here as this strong duality is not used in the present paper. As a consequence, we have the following lemma:

Lemma 2.7. *The locally convex topological vector spaces $L_{loc}^{p,q}(\Omega, F)$ and $L_{cpt}^{p,q}(\Omega, F^*)$ are reflexive.*

Proof. We give another argument to show that the Fréchet space $E := L_{loc}^{p,q}(\Omega, F)$ is reflexive. As $p_1 \leq p_2 \leq p_3 \leq \dots$, we have that $\{p_j\}_j$ is a fundamental system of semi-norms for E . The local Banach spaces $E_j := E / \ker p_j$ are Hilbert-spaces, thus reflexive. So, E is reflexive by [MV], Proposition 25.15. \square

2.6. L^2 -Hilbert space duality. We denote by $*$ the Hodge- $*$ -operator on the complex Hermitian manifold $X^* = X \setminus \text{Sing } X$. It is convenient to work with the conjugate-linear operator

$$*\eta := *\eta.$$

Let $\tau : F \rightarrow F^*$ be the canonical conjugate-linear bundle isomorphism of F onto its dual bundle. We can then define the conjugate-linear isomorphism

$$*_F : \Lambda^{p,q} T^* M \otimes F \rightarrow \Lambda^{n-p,n-q} T^* M \otimes F^*$$

by setting $*_F(\eta \otimes e) := *\eta \otimes \tau(e)$. This gives the following representation for the inner product on (p, q) -forms with values in F on an open set $\Omega \subset X$:

$$(\eta, \psi)_{F, \Omega} = \int_{\Omega^*} \langle \eta, \psi \rangle_F dV_X = \int_{\Omega^*} \eta \wedge *_F \psi, \quad (25)$$

$$\|\eta\|_{F, \Omega} = \sqrt{(\eta, \eta)_{F, \Omega}}. \quad (26)$$

Suppose that η, ψ are smooth forms with values in F and compact support in Ω^* , η of degree $(p, q-1)$ and ψ of degree (p, q) . Then it is easy to compute by Stokes' Theorem that:

$$(\bar{\partial}\eta, \psi)_{F, \Omega} = (-1)^{p+q} \int_{\Omega^*} \eta \wedge \bar{\partial} *_F \psi = - \int_{\Omega^*} \eta \wedge *_F *_F \bar{\partial} *_F \psi = (\eta, -*_F \bar{\partial} *_F \psi)_{F, \Omega}.$$

Thus, we note (cf. e.g. [R4], Lemma 2.2):

Lemma 2.8. *The formal adjoint of the $\bar{\partial}$ -operator for forms with values in the Hermitian holomorphic line bundle F with respect to the $\|\cdot\|_F$ -norm is*

$$\vartheta := -*_F \bar{\partial} *_F. \quad (27)$$

Let

$$\bar{\partial}_{cpt} : A_{cpt}^{p,q}(\Omega^*, F) \rightarrow A_{cpt}^{p,q+1}(\Omega^*, F)$$

be the $\bar{\partial}$ -operator on smooth F -valued forms with compact support in Ω^* . Then we denote by

$$\bar{\partial}_{max} : L^{p,q}(\Omega^*, F) \rightarrow L^{p,q+1}(\Omega^*, F)$$

the maximal and by

$$\bar{\partial}_{min} : L^{p,q}(\Omega^*, F) \rightarrow L^{p,q+1}(\Omega^*, F)$$

the minimal closed Hilbert space extension of the operator $\bar{\partial}_{cpt}$ as densely defined operator from $L^{p,q}(\Omega^*, F)$ to $L^{p,q+1}(\Omega^*, F)$.

For F -valued forms, let $H_{max}^{p,q}(\Omega^*, F)$ be the L^2 -Dolbeault cohomology on Ω^* with respect to the maximal closed extension $\bar{\partial}_{max}$, i.e. the $\bar{\partial}$ -operator in the sense of distributions on Ω^* , and $H_{min}^{p,q}(\Omega^*, F)$ the L^2 -Dolbeault cohomology with respect to the minimal closed extension $\bar{\partial}_{min}$.

Note that the $\bar{\partial}$ -operator in the sense of distributions, $\bar{\partial}_{max}$, agrees with the operator $\bar{\partial}_w$ defined above restricted to $L^{p,q}(\Omega^*, F)$. On the other hand, $\bar{\partial}_{min}$ has the same boundary condition as $\bar{\partial}_s$ at the singular set $\text{Sing } X$, but it comes also with a Dirichlet boundary condition at $b\Omega$ which does not appear for $\bar{\partial}_s$.

We will now identify the Hilbert space adjoints $\bar{\partial}_{max}^*$ and $\bar{\partial}_{min}^*$ of $\bar{\partial}_{max}$ and $\bar{\partial}_{min}$. Let ϑ be the formal adjoint of $\bar{\partial}$ as computed in Lemma 2.8, and denote by ϑ_{cpt} its action on smooth F -valued forms compactly supported in Ω^* :

$$\vartheta_{cpt} : A_{cpt}^{p,q}(\Omega^*, F) \rightarrow A_{cpt}^{p,q-1}(\Omega^*, F).$$

This operator is graph closable as an operator $L_{p,q}^2(\Omega^*, F) \rightarrow L_{p,q-1}^2(\Omega^*, F)$, and as for the $\bar{\partial}$ -operator, we denote by ϑ_{\min} its minimal closed extension, i.e. the closure of the graph, and by ϑ_{\max} the maximal closed extension, that is the ϑ -operator in the sense of distributions with respect to compact subsets of Ω^* . By (27) we have:

$$\vartheta_{\min} = -*_{F^*}\bar{\partial}_{\min}*_{F^*} \quad , \quad \vartheta_{\max} = -*_{F^*}\bar{\partial}_{\max}*_{F^*}.$$

By definition, $\bar{\partial}_{\max} = \vartheta_{\text{cpt}}^*$, and it follows that

$$\bar{\partial}_{\max}^* = (\vartheta_{\text{cpt}}^*)^* = \vartheta_{\text{cpt}} = \vartheta_{\min} = -*_{F^*}\bar{\partial}_{\min}*_{F^*}, \quad (28)$$

where we denote by ϑ_{cpt} also the closure of the graph of ϑ_{cpt} . Analogously, $\vartheta_{\max} = \bar{\partial}_{\text{cpt}}^*$ (by definition) implies

$$\bar{\partial}_{\min}^* = \vartheta_{\max} = -*_{F^*}\bar{\partial}_{\max}*_{F^*}. \quad (29)$$

For the sake of completeness, let us recall (see [R4], Theorem 2.3):

Theorem 2.9. *Assume that the $\bar{\partial}$ -operators in the sense of distributions*

$$\bar{\partial}_{\max} : L^{p,q-1}(\Omega^*, F) \rightarrow L^{p,q}(\Omega^*, F) \quad , \quad \bar{\partial}_{\min} : L^{p,q}(\Omega^*, F) \rightarrow L^{p,q+1}(\Omega^*, F)$$

both have closed range (with the usual assumptions for $q = 0$ or $q = n$). Then there exists a non-degenerate pairing

$$\{\cdot, \cdot\} : H_{\max}^{p,q}(\Omega^*, F) \times H_{\min}^{n-p,n-q}(\Omega^*, F^*) \rightarrow \mathbb{C}$$

given by $\{[\eta], [\psi]\} := \int_{\Omega^} \eta \wedge \psi$.*

2.7. Fréchet sheaves. For convenience of the reader, let us recall from [GR3] a few preliminaries on the (unique) Fréchet space structure on coherent analytic sheaves.

Definition 2.10 ([GR3], Definition VIII.A.3). *Let \mathcal{S} be a sheaf of vector spaces over a topological space X . \mathcal{S} is a **Fréchet sheaf** if there is a neighborhood basis $\mathcal{U} = \{U\}$ of open sets such that the following two conditions hold:*

- (1) *$H^0(U, \mathcal{S})$ can be given the structure of a Fréchet space for all $U \in \mathcal{U}$.*
- (2) *If $U, V \in \mathcal{U}$ and $V \subset U$, then the restriction map $r_{UV} : H^0(U, \mathcal{S}) \rightarrow H^0(V, \mathcal{S})$ is continuous.*

Theorem 2.11 ([GR3], Theorem VIII.A.7). *Let (X, \mathcal{O}_X) be a reduced complex space. There is a unique way of making every coherent analytic sheaf into a Fréchet sheaf so that the following two conditions are satisfied.*

- (1) *If \mathcal{S} is a subsheaf of $(\mathcal{O}_X)^N$, the space of sections of \mathcal{S} has the topology of uniform convergence on compact subsets.*
- (2) *For any two coherent sheaves \mathcal{S}, \mathcal{T} there is a neighborhood basis \mathcal{U} such that $H^0(U, \mathcal{S}), H^0(U, \mathcal{T})$ are Fréchet spaces for all $U \in \mathcal{U}$. Any \mathcal{O}_X -homomorphism $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ is continuous.*

Theorem 2.12 ([GR3], Theorem VIII.A.8). *Let (X, \mathcal{O}_X) be a reduced complex space. Let \mathcal{S} be a coherent analytic sheaf (and thus by Theorem 2.11 a Fréchet sheaf). For any open set W of X , $H^0(W, \mathcal{S})$ can be given the structure of a Fréchet space in a unique way so that the restriction mappings are continuous. We can choose a family of pseudonorms $\|\cdot\|_K$ on $H^0(K, \mathcal{S})$ for all compact sets K , so that, if $K \subset K'$ and $f \in H^0(K', \mathcal{S})$ then $\|f\|_K \leq \|f\|_{K'}$. The topology of $H^0(W, \mathcal{S})$ is that determined by the pseudonorms $\|\cdot\|_K$ for all $K \subset W$.*

3. TOPOLOGICAL SERRE DUALITY FOR $\bar{\partial}$ -OPERATORS

We will now derive duality statements similar to L^2 -Hilbert space duality, Theorem 2.9, for the operators $\bar{\partial}_w$ and $\bar{\partial}_s$ on the spaces $L_{loc}^{p,q}$ and $L_{cpt}^{p,q}$, respectively. We have to face the problem that we deal just with densely defined operators on locally convex topological vector spaces. This requires some extra work. Moreover, the main difficulty is to show that the operators $\bar{\partial}_w$ and $\bar{\partial}_s$ are topologically dual at arbitrary singularities.

3.1. Interpreting $\bar{\partial}_w$ and $\bar{\partial}_s$ as continuous operators. For an open set $\Omega \subset X$, we define the product spaces

$$\begin{aligned} P_{loc}^{p,q}(\Omega, F) &:= L_{loc}^{p,q}(\Omega, F) \times L_{loc}^{p,q+1}(\Omega, F), \\ P_{cpt}^{p,q}(\Omega, F) &:= L_{cpt}^{p,q}(\Omega, F) \times L_{cpt}^{p,q+1}(\Omega, F) \end{aligned}$$

with the product topologies. Note that $P_{loc}^{p,q}(\Omega, F)$ is a Fréchet space and $P_{cpt}^{p,q}(\Omega, F)$ is an (LF)-space in the sense of Dieudonné-Schwartz (see Theorem 2.5).

Here we have continuous shifting operators

$$S : P_{loc/cpt}^{p,q}(\Omega, F) \rightarrow P_{loc/cpt}^{p,q+1}(\Omega, F), \quad (a, b) \mapsto (b, 0),$$

where the index loc/cpt refers to either loc or cpt in the full statement. By Theorem 2.6, there is a non-degenerate pairing

$$P_{loc}^{p,q}(\Omega, F) \times P_{cpt}^{n-p, n-q-1}(\Omega, F^*) \rightarrow \mathbb{C}, \quad (30)$$

$$((\eta_1, \eta_2), (\omega_1, \omega_2)) \mapsto \int_{\Omega^*} \eta_1 \wedge \omega_2 + \int_{\Omega^*} \eta_2 \wedge \omega_1, \quad (31)$$

making $P_{cpt}^{n-p, n-q-1}(\Omega, F^*)$ the topological dual of $P_{loc}^{p,q}(\Omega, F)$ and vice versa.

Consider now the closed densely defined operator

$$\bar{\partial}_w : L_{loc}^{p,q}(\Omega, F) \rightarrow L_{loc}^{p,q+1}(\Omega, F).$$

Let $\Gamma_{w,loc}^{p,q}(\Omega, F)$ be the graph of $\bar{\partial}_w$ in $P_{loc}^{p,q}(\Omega, F)$. The fact that $\bar{\partial}_w$ is a closed operator means nothing else but that $\Gamma_{w,loc}^{p,q}(\Omega, F)$ is a closed subspace of $P_{loc}^{p,q}(\Omega, F)$ making it itself a Fréchet space with the induced topology. As $\bar{\partial}_w \circ \bar{\partial}_w = 0$, we obtain now a new $\bar{\partial}_w$ operator

$$\begin{aligned} \bar{\partial}_w : \Gamma_{w,loc}^{p,q}(\Omega, F) &\rightarrow \Gamma_{w,loc}^{p,q+1}(\Omega, F), \\ (f, \bar{\partial}_w f) &\mapsto (\bar{\partial}_w f, 0), \end{aligned}$$

which is a continuous operator between Fréchet spaces (here, the $\bar{\partial}_w$ -operator is clearly just the restriction of the shift operator: $\bar{\partial}_w = S|_{\Gamma_{w,loc}^{p,q}}$).

Analogously, let $\Gamma_{s,loc}^{p,q}(\Omega, F)$, $\Gamma_{w,cpt}^{p,q}(\Omega, F)$ and $\Gamma_{s,cpt}^{p,q}(\Omega, F)$ be the graphs of the operators $\bar{\partial}_s$, $\bar{\partial}_w$ and $\bar{\partial}_s$ in the spaces $P_{loc}^{p,q}(\Omega, F)$, $P_{cpt}^{p,q}(\Omega, F)$ and $P_{cpt}^{p,q}(\Omega, F)$, respectively. As all the operators under consideration are closed, $\Gamma_{s,loc}^{p,q}(\Omega, F)$ is a Fréchet space, whereas it is straight forward to check that the spaces $\Gamma_{w,cpt}^{p,q}(\Omega, F)$ and $\Gamma_{s,cpt}^{p,q}(\Omega, F)$ are (LF)-spaces in the sense of Dieudonné-Schwartz (see Theorem 2.5).

We obtain continuous linear operators

$$\begin{aligned} \bar{\partial}_s : \Gamma_{s,loc}^{p,q}(\Omega, F) &\rightarrow \Gamma_{s,loc}^{p,q+1}(\Omega, F), \quad (f, \bar{\partial}_s f) \mapsto (\bar{\partial}_s f, 0), \\ \bar{\partial}_w : \Gamma_{w,cpt}^{p,q}(\Omega, F) &\rightarrow \Gamma_{w,cpt}^{p,q+1}(\Omega, F), \quad (f, \bar{\partial}_w f) \mapsto (\bar{\partial}_w f, 0), \\ \bar{\partial}_s : \Gamma_{s,cpt}^{p,q}(\Omega, F) &\rightarrow \Gamma_{s,cpt}^{p,q+1}(\Omega, F), \quad (f, \bar{\partial}_s f) \mapsto (\bar{\partial}_s f, 0), \end{aligned}$$

for which we will study duality relations in the following.

3.2. $\bar{\partial}$ -complexes and their cohomology. For any of the operators defined above, we have $\bar{\partial}_w \circ \bar{\partial}_w = 0$, $\bar{\partial}_s \circ \bar{\partial}_s = 0$ and obtain a $\bar{\partial}$ -complex:

$$0 \rightarrow \Gamma_{w/s, loc/cpt}^{p,0}(\Omega, F) \xrightarrow{\bar{\partial}_{w/s}} \Gamma_{w/s, loc/cpt}^{p,1}(\Omega, F) \xrightarrow{\bar{\partial}_{w/s}} \Gamma_{w/s, loc/cpt}^{p,2}(\Omega, F) \xrightarrow{\bar{\partial}_{w/s}} \dots$$

where one has to choose either w or s in place of w/s and either loc or cpt in place of loc/cpt for the whole statement. Making the connection to Section 2.1, it is easy to see that:

$$\begin{aligned} H^q(\Gamma_{w,loc}^{p,*}(\Omega, F)) &= H^q(\Gamma(\Omega, \mathcal{C}^{p,*}(F))) = H_{w,loc}^{p,q}(\Omega, F), \\ H^q(\Gamma_{s,loc}^{p,*}(\Omega, F)) &= H^q(\Gamma(\Omega, \mathcal{F}^{p,*}(F))) = H_{s,loc}^{p,q}(\Omega, F), \\ H^q(\Gamma_{w,cpt}^{p,*}(\Omega, F)) &= H_{cpt}^q(\Gamma(\Omega, \mathcal{C}^{p,*}(F))) = H_{w,cpt}^{p,q}(\Omega, F), \\ H^q(\Gamma_{s,cpt}^{p,*}(\Omega, F)) &= H_{cpt}^q(\Gamma(\Omega, \mathcal{F}^{p,*}(F))) = H_{s,cpt}^{p,q}(\Omega, F). \end{aligned}$$

In view of duality statements, we have to face the problem that these cohomologies may not be Hausdorff, so we need also the Hausdorff cohomologies defined by the quotient spaces $\ker \bar{\partial}_{w/s} / \overline{\text{Im } \bar{\partial}_{w/s}}$, carrying the quotient topology. We denote these spaces by

$$\overline{H}^q(\Gamma_{w,loc}^{p,*}(\Omega, F)) , \overline{H}^q(\Gamma_{s,loc}^{p,*}(\Omega, F)) , \overline{H}^q(\Gamma_{w,cpt}^{p,*}(\Omega, F)) , H^q(\Gamma_{s,cpt}^{p,*}(\Omega, F)).$$

3.3. Topological Duality between separated $\bar{\partial}_w$ - and $\bar{\partial}_s$ -cohomology. Note that a cohomology class in any of the cohomologies that we consider, $[(f, g)] \in H^q$, is represented by a tuple of forms (f, g) such that $g = \bar{\partial}f = 0$. So, we will just write $[f] \in H^q$ for a cohomology class and f for its representative.

Theorem 3.1. *There is a non-degenerate pairing*

$$H^q(\Gamma_{w,loc}^{p,*}(\Omega, F)) \times H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*)) \rightarrow \mathbb{C} , \quad ([\eta], [\omega]) \mapsto \int_{\Omega^*} \eta \wedge \omega, \quad (32)$$

such that $H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*))$ is the topological dual of $H^q(\Gamma_{w,loc}^{p,*}(\Omega, F))$, and, vice versa, $H^q(\Gamma_{w,loc}^{p,*}(\Omega, F))$ is the topological dual of $H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*))$.

Proof. The integral in (32) exists as $\eta \in L_{loc}^{p,q}$ and $\omega \in L_{cpt}^{n-p, n-q}$. Assume that $\eta = \bar{\partial}_w \eta'$ with $\eta' \in \text{Dom } \bar{\partial}_w \subset L_{loc}^{p, q-1}$. Then

$$\int_{\Omega^*} \eta \wedge \omega = \int_{\Omega^*} \bar{\partial} \eta' \wedge \omega = \int_{\Omega^*} \eta' \wedge \bar{\partial} \omega = 0$$

because ω has compact support in Ω and can be (by definition of $\bar{\partial}_s$) approximated in the graph norm by forms with compact support away from the singular set so that partial integration is possible. Analogously, assume that $\omega = \bar{\partial}_s \omega'$ with $\omega' \in \text{Dom } \bar{\partial}_s \subset L_{cpt}^{n-p, n-q-1}$. Then

$$\int_{\Omega^*} \eta \wedge \omega = \int_{\Omega^*} \eta \wedge \bar{\partial} \omega' = \int_{\Omega^*} \bar{\partial} \eta \wedge \omega' = 0$$

by the same argument for ω' . This shows that the pairing (32) is well-defined.

Let $[\omega] \in H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*))$, represented by $\omega \in \ker \bar{\partial}_s \subset L_{cpt}^{n-p, n-q}(\Omega, F^*)$. Then

$$(\eta, \bar{\partial}_w \eta) \mapsto \int_{\Omega^*} \eta \wedge \omega$$

defines a continuous linear functional on $\Gamma_{w,loc}^{p,q}(\Omega, F)$, and this induces (by the partial integration argument from above) the continuous⁸ linear functional

$$[\eta] \mapsto \int_{\Omega^*} \eta \wedge \omega$$

on the quotient space $H^q(\Gamma_{w,loc}^{p,*}(\Omega, F))$ of the closed subspace $\ker \bar{\partial}_w \subset \Gamma_{w,loc}^{p,q}(\Omega, F)$. Thus, $[\omega]$ represents in fact a continuous linear functional in $(H^q(\Gamma_{w,loc}^{p,*}(\Omega, F)))'$

Conversely, let $\Lambda \in (H^q(\Gamma_{w,loc}^{p,*}(\Omega, F)))'$. As we consider the Hausdorff cohomology, the projection

$$\pi : \ker \bar{\partial}_w \rightarrow \frac{\ker \bar{\partial}_w}{\text{Im } \bar{\partial}_w} = H^q(\Gamma_{w,loc}^{p,*}(\Omega, F))$$

is continuous (see e.g. [R1], Theorem 1.41), and so $\Lambda \circ \pi$ is a continuous linear functional on $\ker \bar{\partial}_w$. But $\ker \bar{\partial}_w$ is a closed subspace of $\Gamma_{w,loc}^{p,q}(\Omega, F)$ which in turn is a closed subspace of the Fréchet space $P_{loc}^{p,q}(\Omega, F)$. So, $\Lambda \circ \pi$ extends by the Hahn-Banach theorem to a continuous linear functional Λ' on $P_{loc}^{p,q}(\Omega, F)$. But we know already that the topological dual space of $P_{loc}^{p,q}(\Omega, F)$ is just $P_{cpt}^{n-p,n-q-1}(\Omega, F^*)$ (see (30), (31)). Thus, there is a tuple $(\lambda_1, \lambda_2) \in P_{cpt}^{n-p,n-q-1}(\Omega, F^*)$ representing Λ' :

$$\begin{aligned} \Lambda' : P_{loc}^{p,q}(\Omega, F) &\rightarrow \mathbb{C}, \\ (f, g) &\mapsto \Lambda'((f, g)) = \int_{\Omega^*} f \wedge \lambda_2 + \int_{\Omega^*} g \wedge \lambda_1. \end{aligned}$$

On $\ker \bar{\partial}_w$, however, we apply Λ' just to tuples of the form $(f, g) = (f, \bar{\partial}_w f) = (f, 0)$. So, the continuous linear functional Λ is simply represented by $\lambda_2 \in L_{cpt}^{n-p,n-q}(\Omega, F^*)$:

$$\Lambda : H^q(\Gamma_{w,loc}^{p,*}(\Omega, F)) \rightarrow \mathbb{C}, \quad [\eta] \mapsto \int_{\Omega^*} \eta \wedge \lambda_2. \quad (33)$$

We claim that $\lambda_2 \in \text{Dom } \bar{\partial}_s$ and $\bar{\partial}_s \lambda_2 = 0$. But (33) implies that

$$(\bar{\partial}_w g, *F^* \lambda_2)_{F, \Omega^*} = \pm \int_{\Omega^*} \bar{\partial}_w g \wedge \lambda_2 = 0$$

for all $g \in \text{Dom } \bar{\partial}_w \subset L_{loc}^{p,q-1}(\Omega, F)$. But then, particularly,

$$(\bar{\partial}_{max} g, *F^* \lambda_2)_{\Omega^*} = 0 \quad \forall g \in \text{Dom } \bar{\partial}_{max} \subset L^{p,q-1}(\Omega^*, F). \quad (34)$$

Now recall that $\bar{\partial}_{max}^* = \vartheta_{min}$ (see (28)). Thus, (34) just means that $\lambda_2 \in \text{Dom } \bar{\partial}_{min} \subset L^{n-p,n-q}(\Omega^*, F^*)$ with $\bar{\partial}_{min} \lambda_2 = 0$. But then we have also $\lambda_2 \in \text{Dom } \bar{\partial}_s$ with $\bar{\partial}_s \lambda_2 = 0$. This shows that in fact any continuous linear functional $\Lambda \in (H^q(\Gamma_{w,loc}^{p,*}(\Omega, F)))'$ is represented by a cohomology class $[\lambda_2] \in H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*))$ under the pairing (32).

It remains to show that conversely $H^q(\Gamma_{w,loc}^{p,*}(\Omega, F))$ is the topological dual of $H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*))$. As above, it is clear that $[\eta] \in H^q(\Gamma_{w,loc}^{p,*}(\Omega, F))$ represents a continuous linear functional in $(H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*)))'$.

So consider $\Lambda \in (H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*)))'$. As above, using the continuous projection $\pi : \ker \bar{\partial}_s \rightarrow \ker \bar{\partial}_s / \text{Im } \bar{\partial}_s$, we obtain the continuous linear functional $\Lambda \circ \pi$

⁸ Let X be a topological vector space, N a closed subspace and X/N the quotient space with the quotient topology. Then the projection $\pi : X \rightarrow X/N$ is an open mapping (see e.g. [R1], Theorem 1.41). So, if Λ is a continuous linear functional on X with $N \subset \ker \Lambda$, then Λ induces a continuous linear functional on X/N .

on $\ker \bar{\partial}_s$ which is a closed subspace of the locally convex topological vector space $P_{cpt}^{n-p, n-q}(\Omega, F^*)$. So, by the Hahn-Banach theorem (see e.g. [R1], Theorem 3.6), there is an extension of $\Lambda \circ \pi$ to a linear continuous functional Λ' on $P_{cpt}^{n-p, n-q}(\Omega, F^*)$ which is then by (30), (31) represented by a tuple $(\mu_1, \mu_2) \in P_{loc}^{p, q-1}(\Omega, F)$:

$$\begin{aligned} \Lambda' : P_{cpt}^{n-p, n-q}(\Omega, F^*) &\rightarrow \mathbb{C}, \\ (f, g) &\mapsto \Lambda'((f, g)) = \int_{\Omega^*} f \wedge \mu_2 + \int_{\Omega^*} g \wedge \mu_1. \end{aligned}$$

On $\ker \bar{\partial}_s$, however, we apply Λ' just to tuples of the form $(f, g) = (f, \bar{\partial}_s f) = (f, 0)$. So, the continuous linear functional Λ is simply represented by $\mu_2 \in L_{loc}^{p, q}(\Omega, F)$:

$$\Lambda : H^{n-q}(\Gamma_{s, cpt}^{n-p, *}(\Omega, F^*)) \rightarrow \mathbb{C}, \quad [\omega] \mapsto \int_{\Omega^*} \omega \wedge \mu_2. \quad (35)$$

We claim that $\mu_2 \in \text{Dom } \bar{\partial}_w$ and $\bar{\partial}_w \mu_2 = 0$, i.e. $\bar{\partial} \mu_2 = 0$ in the sense of distributions on Ω^* . But this is easy to see because (35) implies particularly that

$$\int_{\Omega^*} \bar{\partial} \varphi \wedge \mu_2 = 0 \quad (36)$$

for any smooth testform $\varphi \in A_{cpt}^{n-p, n-q-1}(\Omega^*, F^*)$. This shows that in fact any continuous linear functional $\Lambda \in (H^{n-q}(\Gamma_{s, cpt}^{n-p, *}(\Omega, F^*)))'$ is represented by a cohomology class $[\mu_2] \in H^q(\Gamma_{w, loc}^{p, *}(\Omega, F))$ under the pairing (32). \square

There is another interesting topological duality pairing:

Theorem 3.2. *There is a non-degenerate pairing*

$$H^q(\Gamma_{s, loc}^{p, *}(\Omega, F)) \times H^{n-q}(\Gamma_{w, cpt}^{n-p, *}(\Omega, F^*)) \rightarrow \mathbb{C}, \quad ([\eta], [\omega]) \mapsto \int_{\Omega^*} \eta \wedge \omega, \quad (37)$$

such that $H^{n-q}(\Gamma_{w, cpt}^{n-p, *}(\Omega, F^*))$ is the topological dual of $H^q(\Gamma_{s, loc}^{p, *}(\Omega, F))$, and, vice versa, $H^q(\Gamma_{s, loc}^{p, *}(\Omega, F))$ is the topological dual of $H^{n-q}(\Gamma_{w, cpt}^{n-p, *}(\Omega, F^*))$.

The proof is similar to the proof of Theorem 3.1, but there is an additional difficulty that we should discuss carefully.

Proof. First, it is seen as in the proof of Theorem 3.1 that the pairing (37) is well-defined because partial integration is possible: forms in $\Gamma_{s, loc}^{p, q}(\Omega, F)$ can be approximated in the graph norm by forms with support away from the singular set, and forms in $\Gamma_{w, cpt}^{n-p, n-q}(\Omega, F^*)$ have compact support in Ω .

As in the proof of Theorem 3.1, a cohomology class in $H^{n-q}(\Gamma_{w, cpt}^{n-p, *}(\Omega, F^*))$ defines a continuous linear functional on $H^q(\Gamma_{s, loc}^{p, *}(\Omega, F))$.

Conversely, let $\Lambda \in (H^q(\Gamma_{s, loc}^{p, *}(\Omega, F)))'$. It is seen completely analogous to the proof of Theorem 3.1 that Λ is represented by a form $\mu_2 \in P_{cpt}^{n-p, n-q}(\Omega, F^*)$:

$$\Lambda : H^q(\Gamma_{s, loc}^{p, *}(\Omega, F)) \rightarrow \mathbb{C}, \quad [\eta] \mapsto \int_{\Omega^*} \eta \wedge \mu_2. \quad (38)$$

We claim that $\mu_2 \in \text{Dom } \bar{\partial}_w$ and $\bar{\partial}_w \mu_2 = 0$. But this is the easy case: (38) yields particularly (cf. (36)) that

$$\int_{\Omega^*} \bar{\partial} \varphi \wedge \mu_2 = 0 \quad \forall \varphi \in A_{cpt}^{p, q-1}(\Omega^*, F). \quad (39)$$

It remains to show that $H^q(\Gamma_{s,loc}^{p,*}(\Omega, F))$ is the topological dual of $H^{n-q}(\Gamma_{w,cpt}^{n-p,*}(\Omega, F^*))$. Here, an additional difficulty appears.

Again it is clear that a cohomology class in $H^q(\Gamma_{s,loc}^{p,*}(\Omega, F))$ represents a continuous linear functional on $H^{n-q}(\Gamma_{w,cpt}^{n-p,*}(\Omega, F^*))$.

For the converse, let $\Lambda \in (H^{n-q}(\Gamma_{w,cpt}^{n-p,*}(\Omega, F^*)))'$. It is seen as above that Λ is represented by a form $\lambda_2 \in P_{loc}^{p,q}(\Omega, F)$:

$$\Lambda : H^{n-q}(\Gamma_{w,cpt}^{n-p,*}(\Omega, F^*)) \rightarrow \mathbb{C}, \quad [\omega] \mapsto \int_{\Omega^*} \omega \wedge \lambda_2. \quad (40)$$

We have to show that $\lambda_2 \in \text{Dom } \bar{\partial}_s$ with $\bar{\partial}_s \lambda_2 = 0$. As above, it follows from (40) that

$$\int_{\Omega^*} \bar{\partial} \varphi \wedge \lambda_2 = 0 \quad \forall \varphi \in A_{cpt}^{n-p,n-q-1}(\Omega^*, F^*). \quad (41)$$

Thus $\lambda_2 \in \text{Dom } \bar{\partial}_w$ and $\bar{\partial}_w \lambda_2 = 0$, i.e. $\bar{\partial} \lambda_2 = 0$ in the sense of distributions on Ω^* .

It remains to show that $\lambda_2 \in \text{Dom } \bar{\partial}_s$. That has to be checked on compact subsets $K \subset \subset \Omega$. So, let $K \subset \subset \Omega$ compact and let $\chi \in C_{cpt}^\infty(\Omega)$ be a smooth cut-off function with compact support in Ω which is identically 1 in a neighborhood of K . Then χ and $\bar{\partial} \chi$ are uniformly bounded (in the sup-norm). So, it follows from (40) that

$$\int_{\Omega^*} \bar{\partial}_w(\chi \varphi) \wedge \lambda_2 = 0 \quad (42)$$

for all $\varphi \in \text{Dom } \bar{\partial}_w \subset L^{n-p,n-q-1}(\Omega^*, F^*)$ because then $\chi \varphi \in \Gamma_{w,cpt}^{n-p,n-q-1}(\Omega, F^*)$ and $[\bar{\partial}(\chi \varphi)] = 0 \in H^{n-q}(\Gamma_{w,cpt}^{n-p,*}(\Omega, F^*))$.

But now it follows from (42) that

$$(\bar{\partial}_w \varphi, *_F(\chi \lambda_2))_{\Omega^*, F^*} = \pm \int_{\Omega^*} \bar{\partial}_w \varphi \wedge \chi \lambda_2 = \pm \int_{\Omega^*} \varphi \wedge \bar{\partial} \chi \wedge \lambda_2 \quad (43)$$

$$= \pm (\varphi, *_F(\bar{\partial} \chi \wedge \lambda_2))_{\Omega^*, F^*} \quad (44)$$

for all $\varphi \in \text{Dom } \bar{\partial}_w \subset L^{n-p,n-q-1}(\Omega^*, F^*)$.

Now recall that $\bar{\partial}_w = \bar{\partial}_{max}$ on $L^{n-p,n-q-1}(\Omega^*, F^*)$ and that $\bar{\partial}_{max}^* = \vartheta_{min}$ (see (28)). Thus, (43),(44) just means that $\chi \lambda_2 \in \text{Dom } \bar{\partial}_{min} \subset L^{p,q}(\Omega^*, F)$ with $\bar{\partial}_{min}(\chi \lambda_2) = \bar{\partial} \chi \wedge \lambda_2$. But then we have on K also $\lambda_2 \in \text{Dom } \bar{\partial}_s(K)$ because $\chi \equiv 1$ on K . \square

3.4. The closed range condition. We are clearly interested in replacing the separated cohomology groups $H = \ker / \text{Im}$ in Theorem 3.1 and Theorem 3.2 by the 'real' cohomology $H = \ker / \text{Im}$. So, we need to study closed range conditions for the $\bar{\partial}$ -operators under consideration. As a preparation, let us note:

Lemma 3.3. *Let*

$$\bar{\partial}_{w/s} : \Gamma_{w/s,loc/cpt}^{p,q}(\Omega, F) \longrightarrow \Gamma_{w/s,loc/cpt}^{p,q+1}(\Omega, F)$$

have closed range.

Then the inverse mapping (i.e. the corresponding $\bar{\partial}$ -solution operator)

$$L = (\bar{\partial}_{w/s})^{-1} : \text{Im } \bar{\partial}_{w/s} \longrightarrow \frac{\Gamma_{w/s,loc/cpt}^{p,q}(\Omega, F)}{\ker \bar{\partial}_{w/s}}.$$

is continuous.

Proof. By the open mapping theorem for Fréchet-spaces or for (LF)-spaces (see [DS1], Theorem 1), respectively, we have that

$$\bar{\partial}_{w/s} : \Gamma_{w/s, loc/cpt}^{p,q}(\Omega, F) \longrightarrow \text{Im } \bar{\partial}_{w/s}$$

is a topological homomorphism, i.e. an open mapping. So, the induced bijective mapping

$$\bar{\partial}_{w/s} : \frac{\Gamma_{w/s, loc/cpt}^{p,q}(\Omega, F)}{\ker \bar{\partial}_{w/s}} \longrightarrow \text{Im } \bar{\partial}_{w/s}$$

is also open (and has a continuous inverse mapping). \square

Lemma 3.4. *If $\bar{\partial}_{w/s} : \Gamma_{w/s, loc/cpt}^{p,q}(\Omega, F) \rightarrow \Gamma_{w/s, loc/cpt}^{p,q+1}(\Omega, F)$ has closed range, then*

$$\bar{\partial}_{s/w} : \Gamma_{s/w, cpt/loc}^{n-p, n-q-1}(\Omega, F^*) \rightarrow \Gamma_{s/w, cpt/loc}^{n-p, n-q}(\Omega, F^*)$$

has closed range, too.

Proof. Let $\omega \in \text{Im } \bar{\partial}_{s/w} \subset \Gamma_{s/w, cpt/loc}^{n-p, n-q}(\Omega, F^*)$. Then $\bar{\partial}_{s/w}\omega = 0$. So, ω represents a continuous linear functional on $\Gamma_{w/s, loc/cpt}^{p,q}(\Omega, F)$, and by partial integration one sees that this functional vanishes on $\ker \bar{\partial}_{w/s}$. Thus, ω represents a continuous linear functional

$$\omega : \frac{\Gamma_{w/s, loc/cpt}^{p,q}(\Omega, F)}{\ker \bar{\partial}_{w/s}} \rightarrow \mathbb{C}, \quad [\eta] \mapsto \int_{\Omega^*} \eta \wedge \omega. \quad (45)$$

By continuity of the mapping L from Lemma 3.3, we obtain a continuous linear functional

$$\omega \circ L : \text{Im } \bar{\partial}_{w/s} \rightarrow \mathbb{C}, \quad f \mapsto \int_{\Omega^*} (\bar{\partial}_{w/s})^{-1} f \wedge \omega. \quad (46)$$

By the Hahn-Banach theorem, $\omega \circ L$ extends to a continuous linear functional Λ on $P_{w/s, loc/cpt}^{p,q+1}(\Omega, F)$. As such it is represented by a tuple $(\lambda_1, \lambda_2) \in P_{cpt/loc}^{n-p, n-q-2}(\Omega, F^*)$. On $\text{Im } \bar{\partial}_{w/s} \subset \ker \bar{\partial}_{w/s}$, however, we apply Λ just to tuples of the form $(f, g) = (f, \bar{\partial}_w f) = (f, 0)$. So, we can choose $\lambda_1 = 0$ and the continuous linear functional Λ is simply represented by $\lambda_2 \in L_{cpt/loc}^{n-p, n-q-1}(\Omega, F^*)$ (see (33), (35), (38) or (40), respectively).

We claim that $\lambda_2 \in \text{Dom } \bar{\partial}_{s/w}$ with $\bar{\partial}_{s/w}\lambda_2 = \omega$. To see that, note that it follows from (45) and (46) that

$$\int_{\Omega^*} \eta \wedge \omega = \int_{\Omega^*} \bar{\partial}_{w/s} \eta \wedge \lambda_2$$

for all $\eta \in \text{Dom } \bar{\partial}_{w/s} \subset L_{loc/cpt}^{p,q}(\Omega, F)$.

For $\lambda_2 \in \text{Dom } \bar{\partial}_s \subset L_{cpt}^{n-p, n-q-1}(\Omega, F^*)$, the claim follows as in Theorem 3.1, (34).

For $\lambda_2 \in \text{Dom } \bar{\partial}_w \subset L_{loc}^{n-p, n-q-1}(\Omega, F^*)$, proceed as in Theorem 3.1, (36).

For $\lambda_2 \in \text{Dom } \bar{\partial}_w \subset L_{cpt}^{n-p, n-q-1}(\Omega, F^*)$, proceed as in Theorem 3.2, (39).

In the most difficult case $\lambda_2 \in \text{Dom } \bar{\partial}_s \subset L_{loc}^{n-p, n-q-1}(\Omega, F^*)$, proceed as in the proof of Theorem 3.2, (41) – (43). \square

3.5. The duality theorem and proof of Theorem 1.4. Summarizing, we obtain from Theorem 3.1, Theorem 3.2 and Lemma 3.4 the following duality theorem:

Theorem 3.5. *Let X be a (singular) Hermitian complex space of pure dimension n , $\Omega \subset X$ an open set and $F \rightarrow \Omega^*$ a Hermitian holomorphic line bundle.*

Let $0 \leq p, q \leq n$ and assume that either both,

$$\bar{\partial}_w : \Gamma_{w,loc}^{p,q-1}(\Omega, F) \rightarrow \Gamma_{w,loc}^{p,q}(\Omega, F), \quad \bar{\partial}_w : \Gamma_{w,loc}^{p,q}(\Omega, F) \rightarrow \Gamma_{w,loc}^{p,q+1}(\Omega, F) \quad (47)$$

or both,

$$\bar{\partial}_s : \Gamma_{s,cpt}^{n-p,n-q-1}(\Omega, F^*) \rightarrow \Gamma_{s,cpt}^{n-p,n-q}(\Omega, F^*), \quad \bar{\partial}_s : \Gamma_{s,cpt}^{n-p,n-q}(\Omega, F^*) \rightarrow \Gamma_{s,cpt}^{n-p,n-q+1}(\Omega, F^*)$$

have closed range (with the usual conventions for $q = 0$ or $q = n$).

Then there is a non-degenerate pairing

$$H^q(\Gamma_{w,loc}^{p,*}(\Omega, F)) \times H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*)) \rightarrow \mathbb{C}, \quad ([\eta], [\omega]) \mapsto \int_{\Omega^*} \eta \wedge \omega, \quad (48)$$

such that $H^{n-q}(\Gamma_{s,cpt}^{n-p,}(\Omega, F^*))$ is the topological dual of $H^q(\Gamma_{w,loc}^{p,*}(\Omega, F))$, and, vice versa, $H^q(\Gamma_{w,loc}^{p,*}(\Omega, F))$ is the topological dual of $H^{n-q}(\Gamma_{s,cpt}^{n-p,*}(\Omega, F^*))$.*

The same statement holds with all indices $\{w, s\}$ replaced by $\{s, w\}$. There is then a non-degenerate topological pairing

$$H^q(\Gamma_{s,loc}^{p,*}(\Omega, F)) \times H^{n-q}(\Gamma_{w,cpt}^{n-p,*}(\Omega, F^*)) \rightarrow \mathbb{C}, \quad ([\eta], [\omega]) \mapsto \int_{\Omega^*} \eta \wedge \omega. \quad (49)$$

To prove Theorem 1.4, assume that $H_{w,loc}^{p,q}(\Omega, F)$ and $H_{w,loc}^{p,q+1}(\Omega, F)$ are Hausdorff. Then the operators in (47) have closed range, and so the first part of Theorem 1.4 follows directly from Theorem 3.5. The second statement of Theorem 1.4 is proven analogously.

3.6. Topology of compact convergence on canonical sheaves. Recall the canonical sheaves $\mathcal{K}_X = \ker \bar{\partial}_w \subset \mathcal{C}^{n,0}$, $\mathcal{K}_X^s = \ker \bar{\partial}_s \subset \mathcal{F}^{n,0}$. So, for any open set $\Omega \subset X$,

$$\mathcal{K}_X(\Omega) = \ker \bar{\partial}_w \subset \Gamma_{w,loc}^{n,0}(\Omega)$$

and

$$\mathcal{K}_X^s(\Omega) = \ker \bar{\partial}_s \subset \Gamma_{s,loc}^{n,0}(\Omega)$$

carry the induced Fréchet space structure of L^2 -convergence on compact subsets of Ω . It is clear that restriction maps are continuous, and so \mathcal{K}_X and \mathcal{K}_X^s are Fréchet sheaves according to Definition 2.10.

On the other hand, the Grauert-Riemenschneider canonical sheaf \mathcal{K}_X is a coherent analytic sheaf. So, \mathcal{K}_X carries also the unique Fréchet sheaf structure of uniform convergence on compact subsets according to Theorem 2.11 and Theorem 2.12. For an open set $\Omega \subset X$, let $\mathcal{K}_X(\Omega)$ carry the Fréchet space structure of L^2 -convergence on compact subsets, and $\widehat{\mathcal{K}}_X(\Omega)$ be the same algebraic vector space with the Fréchet space structure of uniform convergence on compact subsets. Then the identity map

$$\widehat{\mathcal{K}}_X(\Omega) \longrightarrow \mathcal{K}_X(\Omega)$$

is bijective and continuous. So, the two Fréchet space structures coincide by the open mapping theorem. To be more precise: It is enough to study the question locally (i.e. for small Ω). So, consider an epimorphism $\beta : \mathcal{O}_X^N(\Omega) \rightarrow \mathcal{K}_X(\Omega)$, where \mathcal{O}_X^N carries the unique Fréchet sheaf structure of uniform convergence on compact subsets.

This map β is clearly continuous. So, by the open mapping theorem, the Fréchet space structures of $\mathcal{K}_X(\Omega)$ and $\mathcal{O}_X^N(\Omega)/\ker \beta$ coincide. But $\mathcal{O}_X^N(\Omega)/\ker \beta \cong \widehat{\mathcal{K}_X}(\Omega)$ topologically by Theorem 2.11.

Concerning the canonical sheaf with boundary condition, \mathcal{K}_X^s , the same argument holds as soon as we knew that the sheaf is coherent. By now, this is just proven in the case of isolated singularities (see [R4], Theorem 1.10). So, we summarize:

Theorem 3.6. *Let X be a Hermitian complex space. Then, on the Grauert-Riemenschneider canonical sheaf \mathcal{K}_X , the Fréchet sheaf structure of L^2 -convergence on compact subsets and the Fréchet sheaf structure of uniform convergence on compact subsets coincide.*

If X has only isolated singularities, then the same holds for the canonical sheaf with Dirichlet boundary condition, \mathcal{K}_X^s .

3.7. Topological equivalence of L^2 -cohomology and Čech cohomology. Let X be a reduced complex space. Then any coherent analytic sheaf $\mathcal{S} \rightarrow X$ carries a unique canonical Fréchet sheaf structure (with the topology of uniform compact convergence, see Section 2.7).

By taking a Leray covering, this induces a canonical Fréchet space topology on the Čech cohomology groups $\check{H}^q(\Omega, \mathcal{S})$ for $\Omega \subset X$ open. We will now show that this Fréchet space topology coincides with our L^2 -topology when we have a suitable Dolbeault isomorphism.

The central tool needed for that is the following statement:

Lemma 3.7. *Let A^* and B^* be two complexes of Fréchet spaces (or of (LF) -spaces, respectively) with continuous linear differentials, and let $u : A^* \rightarrow B^*$ be a continuous linear morphism from A^* to B^* . If u is an algebraic quasi-isomorphism, i.e., if it induces an algebraic isomorphism of the cohomology groups of the complexes, then u is also a topological quasi-isomorphism, i.e. it induces a topological isomorphism of the cohomology groups carrying their natural induced topology.*

Proof. For complexes of Fréchet spaces, the statement is just [RR], Lemma 1. But the proof in [RR] holds literally also for complexes of (LF) -spaces as the open mapping theorem is valid for such spaces. \square

Let X be a (possibly singular) Hermitian complex space (this includes particularly the case of a Hermitian complex manifold) and consider the fine resolution

$$0 \rightarrow \mathcal{K}_X \longrightarrow \mathcal{C}^{n,0} \xrightarrow{\bar{\partial}_w} \mathcal{C}^{n,1} \xrightarrow{\bar{\partial}_w} \mathcal{C}^{n,2} \xrightarrow{\bar{\partial}_w} \dots$$

according to Theorem 2.3. Let $\Omega \subset X$ be an open set. Then the abstract DeRham-isomorphism

$$\check{H}^q(\Omega, \mathcal{K}_X) \cong H^q(\Gamma(\Omega, \mathcal{C}^{n,*})) = H^q(\Gamma_{w,loc}^{n,*}(\Omega)) = H_{w,loc}^{n,q}(\Omega) \quad (50)$$

can be realized explicitly (cf. [D], IV.6 for the following). Let $\mathcal{U} = \{U_\alpha\}$ be a Leray covering for Ω , and let $\{\chi_\alpha\}$ be a smooth partition of unity subordinate to \mathcal{U} . Given a Čech cocycle $c \in C^q(\mathcal{U}, \mathcal{K}_X)$, we define a Čech cocycle $f \in C^0(\mathcal{U}, \mathcal{C}^{n,q})$ by

$$f_\alpha := \sum_{\nu_0, \dots, \nu_{q-1}} \bar{\partial} \chi_{\nu_0} \wedge \dots \wedge \bar{\partial} \chi_{\nu_{q-1}} \cdot c_{\nu_0 \dots \nu_{q-1}} \alpha \quad \text{on } U_\alpha.$$

In fact, f is a cocycle and defines a $\bar{\partial}$ -closed global section

$$\phi_c = \sum_{\nu_q} \chi_{\nu_q} f_{\nu_q} = \sum_{\nu_0, \dots, \nu_q} \chi_{\nu_q} \bar{\partial} \chi_{\nu_0} \wedge \dots \wedge \bar{\partial} \chi_{\nu_{q-1}} \cdot c_{\nu_0 \dots \nu_{q-1} \nu_q} \in \Gamma_{w,loc}^{n,q}(\Omega).$$

This mapping

$$\Psi : C^q(\mathcal{U}, \mathcal{K}_X) \longrightarrow \Gamma_{w,loc}^{n,q}(\Omega), \quad c \mapsto \phi_c,$$

is continuous as $C^q(\mathcal{U}, \mathcal{K}_X)$ carries the Fréchet space structure of uniform convergence and $\Gamma_{w,loc}^{n,q}(\Omega)$ the Fréchet space structure of L^2 -convergence (the contribution of the partition of unity is harmless).

Taking into account also the topological isomorphism $\check{H}^q(\Omega, \mathcal{K}_X) \cong \check{H}^q(\mathcal{U}, \mathcal{K}_X)$, the algebraic isomorphism (50) is then realized by the induced mapping

$$[\Psi] : \check{H}^q(\mathcal{U}, \mathcal{K}_X) \xrightarrow{\cong} H_{w,loc}^{n,q}(\Omega), \quad [c] \mapsto [\phi_c].$$

But now $[\Psi]$ is also a topological isomorphism by Lemma 3.7 (recall that the $\bar{\partial}$ -operators are continuous in our Γ -complexes).

Verbatim the same argument can be applied to \mathcal{K}_X^s if X has only isolated singularities (see Theorem 2.4). Summing up, we get:

Theorem 3.8. *Let X be a possibly singular Hermitian complex space, $\Omega \subset X$ open. Then the Dolbeault isomorphism induces a topological isomorphism*

$$\check{H}^q(\Omega, \mathcal{K}_X) \xrightarrow{\cong} H^q(\Gamma_{w,loc}^{n,*}(\Omega)) = H_{w,loc}^{n,q}(\Omega),$$

where the Čech cohomology carries the canonical Fréchet space structure of uniform convergence on compact subsets and $H_{w,loc}^{n,q}(\Omega)$ the Fréchet space structure of L^2 -convergence on compact subsets.

If X has only isolated singularities, then the analogous statement holds for the Dolbeault isomorphism

$$\check{H}^q(\Omega, \mathcal{K}_X^s) \xrightarrow{\cong} H^q(\Gamma_{s,loc}^{n,*}(\Omega)) = H_{s,loc}^{n,q}(\Omega).$$

It is just for ease of notation that we did not incorporate a Hermitian line bundle $F \rightarrow X$ in this section and the last one. Note that all that has been said holds as well for forms with values in such a line bundle.

It also remains to show that the algebraic isomorphisms (18), (19), (21) and (22) from Section 2.3 are topological isomorphisms, too. Let us just show this for (18) and the trivial bundle F , i.e., for

$$H_{w,loc}^{n,q}(\Omega) = H^q(\Gamma_{w,loc}^{n,*}(\Omega)) \xrightarrow{\cong} H^q(\Gamma_{w,loc}^{n,*}(\pi^{-1}(\Omega))) = H_{w,loc}^{n,q}(\pi^{-1}(\Omega)). \quad (51)$$

The other statements are shown completely analogous (and the line bundles do not matter at all). We have to consider the continuous linear morphism of complexes

$$\pi^* : (\Gamma_{w,loc}^{n,*}(\Omega), \bar{\partial}_w) \longrightarrow (\Gamma_{w,loc}^{n,*}(\pi^{-1}(\Omega)), \bar{\partial}_w)$$

which induces the algebraic isomorphism (51). Hence, (51) is also a topological isomorphism by Lemma 3.7, and we note:

Lemma 3.9. *The algebraic isomorphisms (18), (19), (21) and (22) from Section 2.3 are also topological isomorphisms.*

3.8. Examples of separated cohomology groups and proof of Theorem 1.5.

If Z is a compact complex space, then the Cartan-Serre theorem shows that the cohomology of coherent analytic sheaves on Z is finite-dimensional, in particular Hausdorff. More generally, we get the Hausdorff property if Z is holomorphically convex: in that case $\check{H}^*(Z, \mathcal{S})$ is Hausdorff for any coherent analytic sheaf \mathcal{S} by [P], Lemma II.1 (to conclude the statement from [P], recall that a holomorphically convex space Z has a (proper) Remmert reduction $\pi : Z \rightarrow Y$ such that Y is Stein).

Let us now prove Theorem 1.5. Let X be a Hermitian complex space, $\dim X = n$, $\Omega \subset X$ holomorphically convex and $\pi : M \rightarrow X$ a resolution of singularities. Then $\pi^{-1}(\Omega)$ is again holomorphically convex and it follows as explained above that

$$\check{H}^q(\Omega, \mathcal{K}_X) , \check{H}^q(\Omega, \mathcal{K}_X^s) , \check{H}^{n-q}(\pi^{-1}(\Omega), \mathcal{O}_M)$$

are Hausdorff for all $0 \leq q \leq n$ (for \mathcal{K}_X^s , assume that X has only isolated singularities so that \mathcal{K}_X^s is coherent).

So, it follows from Theorem 3.8 (which holds as well for the cohomology with values in \mathcal{O}_M on the smooth manifold M) that the L^2 -cohomology groups

$$H_{w,loc}^{n,q}(\Omega) , H_{s,loc}^{n,q}(\Omega) , H_{w,loc}^{0,n-q}(\pi^{-1}(\Omega)) \quad (52)$$

are Hausdorff for all $0 \leq q \leq n$.

Thus, by Lemma 3.4,

$$H_{w,cpt}^{n,q}(\pi^{-1}(\Omega)) = H_{s,cpt}^{n,q}(\pi^{-1}(\Omega)) \quad (53)$$

is Hausdorff for all $0 \leq q \leq n$ ($\bar{\partial}_w$ and $\bar{\partial}_s$ coincide on M which has no singular set).

But now Lemma 3.9 yields that

$$H_{w,cpt}^{n,q}(\Omega) , H_{s,cpt}^{n,q}(\Omega) \quad (54)$$

are Hausdorff for all $0 \leq q \leq n$ (here, for the statement about the $\bar{\partial}_s$ -cohomology, we have to assume that X has only isolated homogeneous singularities, see (22)).

From (52) and (54) we see that actually – as claimed – all the $H^{n,q}$ -cohomology groups in the statement of Theorem 1.5 are Hausdorff (under the conditions imposed on the singularities of X for the $\bar{\partial}_s$ -cohomology).

The dual $H^{0,n-q}$ -cohomology groups are then Hausdorff by Lemma 3.4 and that completes the proof of Theorem 1.5.

There is another interesting example of Hausdorff cohomology. Let $\Omega \subset X$ be q -convex. Then it follows by the Andreotti-Grauert theory that $H^r(\Omega, \mathcal{S})$ is of finite dimension (hence Hausdorff) for any coherent analytic sheaf \mathcal{S} and all $r \geq q$.

4. VANISHING THEOREMS FOR THE $\bar{\partial}$ -COHOMOLOGY ON SINGULAR SPACES

4.1. Proof of Theorem 1.6. Both statements follow simply from the combination of Theorem 3.8, Lemma 3.4 and Theorem 3.5.

4.2. On the domain of the $\bar{\partial}_s$ -operator. In view of our vanishing result, Theorem 1.6, it would be good to understand the $\bar{\partial}_s$ -operator better, but, in general, it is difficult to decide if a differential form is in the domain of the $\bar{\partial}_s$ -operator. However, we have at least the following criterion:

Theorem 4.1. *Locally bounded forms in the domain of the $\bar{\partial}_w$ -operator are also in the domain of the $\bar{\partial}_s$ -operator.*

Here, locally bounded means bounded on $K^* = K - \text{Sing } X$ for compact sets $K \subset\subset X$. A proof of Theorem 4.1, using resolution of singularities, was given in [R5], Theorem 1.6. But Theorem 4.1 can be achieved also in a more direct way as an easy consequence of Sibony's results on extension of analytic objects [S1]. The following short argument was communicated to the author by Nessim Sibony.

Proof. As the problem is local, consider a Hermitian complex space X of dimension n , embedded locally in some open set $\Omega \subset \mathbb{C}^N$. Let T be the positive closed current of integration over X , and A the singular set of X (as a subset of Ω), which is a complete pluripolar set. By [S1], Lemma 1.2, there exists a sequence $\{u_j\}$ of smooth plurisubharmonic functions, $0 \leq u_j \leq 1$, vanishing identically on a neighborhood of A , converging uniformly to $1_{\Omega \setminus A}$ on compact subsets of $\Omega \setminus A$.

Consider now compact sets $K, L \subset \Omega$ with $K \subset \text{int}(L)$, and let $0 \leq \chi \leq 1$ be a test function with support in L and $\chi = 1$ on K . Let $\omega := i\partial\bar{\partial}\|z\|^2$ be the Kähler form in \mathbb{C}^N . Then, similarly as in the proof of Theorem 1.3 in [DS2], p. 361, we have

$$\begin{aligned} \|\bar{\partial}u_j\|_{L^2(K \cap X)}^2 &\leq \int \chi^2 i\partial u_j \wedge \bar{\partial}u_j \wedge T \wedge \omega^{n-1} \leq \frac{1}{2} \int \chi^2 i\partial\bar{\partial}u_j^2 \wedge T \wedge \omega^{n-1} \\ &= \frac{1}{2} \int \chi^2 i\partial\bar{\partial}(u_j^2 - 1) \wedge T \wedge \omega^{n-1} = \frac{1}{2} \int (u_j^2 - 1) i\partial\bar{\partial}\chi^2 \wedge T \wedge \omega^{n-1} \\ &\leq C_\chi \|u_j^2 - 1\|_{L^1(L \cap X)} \xrightarrow{j \rightarrow \infty} 0, \end{aligned}$$

where we have convergence to 0 because the u_j^2 converge uniformly to 1 on compact subsets of $\Omega \setminus A$.

So, let f be a bounded form in the domain of $\bar{\partial}_w$. Then it is easy to see (using the Hölder inequality) that $f_j := u_j f$ is a sequence as required in (9), (10). \square

4.3. Hartogs' extension theorem. Let X be a connected normal complex space of dimension $n \geq 2$ which is cohomologically $(n-1)$ -complete. Then

$$H_{s, \text{cpt}}^{0,1}(X) = H^1(\Gamma_{s, \text{cpt}}^{0,*}(X)) = 0 \quad (55)$$

by Theorem 1.6. As in [H2], Section 2.3, one can now use (55) to give a short proof of Hartogs' extension theorem in its most general form by the $\bar{\partial}$ -method of Ehrenpreis. We repeat the argument for convenience of the reader.

Let D be a domain in X and $K \subset D$ a compact subset such that $D \setminus K$ is connected. Let $f \in \mathcal{O}(D \setminus K)$. We claim that f has a unique holomorphic extension to the whole domain D .

To show that, let $\chi \in C_{\text{cpt}}^\infty(X)$ be a smooth cut-off function that is identically 1 in a neighborhood of K such that $C := \text{supp } \chi \subset\subset D$.

Consider

$$g := (1 - \chi)f \in C^\infty(D),$$

which is an extension of f to D , but unfortunately not holomorphic. We have to fix it by the idea of Ehrenpreis. So, let

$$\omega := \bar{\partial}g,$$

which is a bounded $\bar{\partial}_w$ -closed $(0,1)$ -form with compact support in D . By Theorem 4.1, ω is also $\bar{\partial}_s$ -closed. We may consider ω as an L^2 -form on X with compact support.

But $H_{s,cpt}^{0,1}(X) = 0$ as seen above, (55). So, there exists $h \in L_{cpt}^{0,0}(X^*)$ such that

$$\bar{\partial}_s h = \omega,$$

and h is holomorphic on $X^* \setminus C$ (where $\omega \equiv 0$). Since X , being $(n-1)$ -complete, is non-compact, it follows by standard arguments (involving the identity theorem) that $h \equiv 0$ on an open subset of $D \setminus C \subset D \setminus K$. Note that X is normal, thus locally irreducible, and so connected subsets of X satisfy the identity theorem.

Let

$$F := (1 - \chi)f - h \in \mathcal{O}(X^*).$$

As X is normal, F extends by the Riemann extension theorem to a holomorphic function on X , say $F \in \mathcal{O}(X)$ for ease of notation (see e.g. [GR1], Chapter 7.4.2). As h is vanishing on an open subset of $D \setminus K$, F is the desired unique extension by the identity theorem.

4.4. Solution of the $\bar{\partial}$ -equation for bounded forms with compact support.

By the method applied in the last section, we obtain also:

Theorem 4.2. *Let X be a Hermitian complex space of pure dimension n , $F \rightarrow X$ a Hermitian holomorphic line bundle, $\Omega \subset X$ a cohomologically q -complete subset. Let $1 \leq r \leq n - q$ and f an F -valued bounded $(0, r)$ -form on Ω^* with compact support in Ω that is $\bar{\partial}$ -closed in the sense of distributions. Then there exists an F -valued L^2 -form with compact support in Ω , $h \in L_{cpt}^{0, r-1}(\Omega, F)$, s.t. $\bar{\partial}h = f$ in the sense of distributions on Ω^* .*

Proof. Just combine Theorem 1.6 and Theorem 4.1. □

For $(0, 1)$ -forms, we can say more, using a resolution of singularities:

Theorem 4.3. *In the situation of Theorem 4.2, let Ω be cohomologically $(n-1)$ -complete and f a bounded $(0, 1)$ -form. Then the solution h can be chosen to be a bounded function on Ω^* . We write for that:*

$$H_{L^\infty, cpt}^{0,1}(\Omega^*, F) = 0.$$

Proof. Consider a resolution of singularities $\pi : \Omega' \rightarrow \Omega$. Then π^*f is a bounded $(0, 1)$ -form and $\bar{\partial}$ -closed in the sense of distributions by [R2], Theorem 3.2 (the $\bar{\partial}$ -equation extends over the exceptional set). But $H_{cpt}^{0,1}(\Omega', \pi^*F) = 0$ by [CR], Theorem 2.6. So, there exists a bounded function h with compact support and $\bar{\partial}h = \pi^*f$. Pushing forward h outside the exceptional set gives the desired bounded solution. □

5. L^2 -COHOMOLOGY AND RATIONAL SINGULARITIES

5.1. Proof of Theorem 1.1. Irreducible components of different dimension can be treated separately, so we can assume that X is of pure dimension n .

Let $\Omega \subset X$ be holomorphically convex. By Theorem 1.5 (and its proof) the cohomology groups

$$H_{s, loc}^{0, q}(\Omega), H_{w, cpt}^{n, n-q}(\Omega), H_{w, loc}^{0, q}(\pi^{-1}(\Omega)), H_{w, cpt}^{n, n-q}(\pi^{-1}(\Omega))$$

are Hausdorff for all $0 \leq q \leq n$.

So, there exist by Theorem 1.4 non-degenerate topological pairings (recall that $\bar{\partial}_w$ and $\bar{\partial}_s$ coincide on M as there are no singularities at all)

$$H_{s,loc}^{0,q}(\Omega) \times H_{w,cpt}^{n,n-q}(\Omega) \longrightarrow \mathbb{C} \quad (56)$$

and

$$H_{w,loc}^{0,q}(\pi^{-1}(\Omega)) \times H_{w,cpt}^{n,n-q}(\pi^{-1}(\Omega)) \longrightarrow \mathbb{C}. \quad (57)$$

But π induces by pull-back of (n, q) -forms natural topological isomorphisms

$$[\pi^*] : H_{w,cpt}^{n,n-q}(\Omega) \xrightarrow{\cong} H_{w,cpt}^{n,n-q}(\pi^{-1}(\Omega))$$

(see (19) and Lemma 3.9). So, (56) and (57) induce dual topological isomorphisms

$$H^q(\pi^{-1}(\Omega), \mathcal{O}_M) \cong H_{w,loc}^{0,q}(\pi^{-1}(\Omega)) \xrightarrow{\cong} H_{s,loc}^{0,q}(\Omega)$$

for all $0 \leq q \leq n$.

5.2. Proofs of Theorem 1.2 and of Theorem 1.3. Both statements are simple corollaries of Theorem 1.1. For the proof of Theorem 1.2, just recall that

$$(\mathcal{H}^0(\mathcal{F}^{0,*}))_x = \varinjlim_{x \in \Omega} H_{s,loc}^{0,q}(\Omega)$$

and

$$(R^q \pi_* \mathcal{O}_M)_x = \varinjlim_{x \in \Omega} H^q(\pi^{-1}(\Omega), \mathcal{O}_M).$$

Recall also that $(\pi_* \mathcal{O}_M)_x = \widehat{\mathcal{O}}_{X,x}$, where $\widehat{\mathcal{O}}_X$ denotes the sheaf of weakly holomorphic functions.

Theorem 1.3 is now in turn a simple corollary of Theorem 1.2, keeping in mind that a point $x \in X$ is by definition rational if it is normal and $(R^q \pi_* \mathcal{O}_M)_x = 0$ for all $q > 0$.

5.3. Homogeneous isolated singularities. For the sake of completeness, let us include also the following observation:

Theorem 5.1. *Let X be a Hermitian complex space with only homogeneous (conical) isolated singularities, $\pi : M \rightarrow X$ a resolution of singularities and $\Omega \subset X$ holomorphically convex. Then push-forward of forms induces for all $q \geq 0$ a natural topological isomorphism*

$$H^q(\pi^{-1}(\Omega), \mathcal{O}_M) \xrightarrow{\cong} H_{w,loc}^{0,q}(\Omega).$$

The L^2 - $\bar{\partial}$ -complex

$$0 \rightarrow \mathcal{O}_X \longrightarrow \mathcal{C}^{0,0} \xrightarrow{\bar{\partial}_w} \mathcal{C}^{0,1} \xrightarrow{\bar{\partial}_w} \mathcal{C}^{0,2} \xrightarrow{\bar{\partial}_s} \mathcal{C}^{0,3} \xrightarrow{\bar{\partial}_w} \dots \quad (58)$$

is exact in a point $x \in X$ if and only if x is a rational point.

Proof. If X has only homogeneous isolated singularities, then $\mathcal{K}_X \cong \mathcal{K}_X^s$ by [R4], Theorem 1.10, with $D = \emptyset$ because homogeneous isolated singularities can be resolved by a single blow-up. So, the statements can be seen completely analogous to the proofs of Theorem 1.1 and of Theorem 1.3.

Note that clearly $\mathcal{O}_{X,x} \subset \widehat{\mathcal{O}}_{X,x} \subset (\ker \bar{\partial}_w^{0,0})_x$. Moreover, $(\ker \bar{\partial}_w^{0,0})_x \subset \mathcal{O}_{X,x}$ if X is normal by the Riemann Extension Theorem for singular spaces (by normality, the codimension of the singular set is bigger than one). So, a point $x \in X$ is normal exactly if $(\ker \bar{\partial}_w^{0,0})_x = \mathcal{O}_{X,x}$. \square

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